JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **36**, No. 1, pp. 102-118

doi: 10.4208/jpde.v36.n1.7
March 2023

Serrin-Type Overdetermined Problem in $\mathbb{H}^n$

GAO Zhenghuan[1], JIA Xiaohan[2] and YAN Jin[3,*]

[1] *Department of Mathematics, Shanghai University, Shanghai 200444, China.*
[2] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, China.*
[3] *School of Mathematical Sciences, University of Science and Technology of China, Hefei 230025, China.*

Received 16 January 2022; Accepted 11 March 2022

Abstract. In this paper, we prove the symmetry of the solution to overdetermined problem for the equation $\sigma_k(D^2u-uI)=C_n^k$ in hyperbolic space. Our approach is based on establishing a Rellich-Pohozaev type identity and using a P function. Our result generalizes the overdetermined problem for Hessian equation in Euclidean space.

AMS Subject Classifications: 35N25, 35B06, 35B50, 53C24

Chinese Library Classifications: O175.25

Key Words: Overdetermined problems; hyperbolic space; P functions; Rellich-Pohozaev identity.

1 Introduction

In the seminal paper [1] Serrin established the symmetry of the solution to

$$\Delta u = n \tag{1.1}$$

in a bounded C^2 domain $\Omega \subset \mathbb{R}^n$ with

$$u=0 \quad \text{and} \quad u_\gamma = 1 \quad \text{on } \partial\Omega, \tag{1.2}$$

where γ is the unit outer normal to $\partial\Omega$. If $u \in C^2(\overline{\Omega})$ is a solution to (1.1) and (1.2), then $u=\frac{|x|^2-1}{2}$ upto a translation and Ω is the unitary ball. The proof is based on the method of *moving planes* and it can be applied to more general uniformly elliptic equations. In [2] Weinberger provided an alternative proof by using maximum principle for P function and a Rellich-Pohozaev type identity.

*Corresponding author. *Email addresses:* gzh@shu.edu.cn (Z. Gao), jiaxiaohan@xmu.edu.cn (X. Jia), yjorac-le@mail.ustc.edu.cn (J. Yan)

There have been many generalizations of Serrin and Weinberger's work to quasilinear elliptic equations (see, e.g., [3–5] and reference therein) and fully nonlinear equations such as Hessian equation and Weingarten curvature equation (see, e.g., [6–8]). In Euclidean space, the overdetermined boundary problem for $\sigma_k(D^2u)=C_n^k$ was studied in [6] by using a Rellich-Pohozaev type identity and some geometric inequalities and was also dealt in [8] by using method of moving planes. Using the P function $P=|Du|^2-2u$ as mentioned in [2,9] we can give an alternative proof which is parallel to Weinberger's.

Theorem 1.1 ([6]). *Suppose $\Omega\subset\mathbb{R}^n$ is a C^2 bounded domain and $u\in C^3(\Omega)\cap C^2(\overline{\Omega})$ is a solution to the following problem*

$$\begin{cases}\sigma_k(D^2u)=C_n^k & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\\ u_\gamma=c_0 & \text{on } \partial\Omega,\end{cases} \tag{1.3}$$

with $k\in\{1,\cdots,n\}$ and c_0 a positive constant. Then upto a translation, $u=\frac{|x|^2-1}{2}$ and Ω is a ball of radius c_0.

In space forms, a few work has been done to generalize the Serrin's symmetry to equation $\Delta u+nKu=c$ using the method of moving planes or P functions and Rellich-Pohozaev type identities (see [10–14] and reference therein).

The hyperbolic space $\mathbb{H}^n$ can be described as the warped product space $[0,\infty)\times\mathbb{S}^{n-1}$ equipped with the rotationally symmetric metric

$$g=dr^2+h^2g_{\mathbb{S}^{n-1}}, \tag{1.4}$$

where $h=\sinh r$, $g_{\mathbb{S}^{n-1}}$ is the round metric on the $n-1$ dimensional sphere.

In the present paper, we consider the overdetermined problem below in hyperbolic space,

$$\begin{cases}\sigma_k(D^2u-uI)=C_n^k & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\\ u_\gamma=c_0 & \text{on } \partial\Omega,\end{cases} \tag{1.5}$$

where Ω is a bounded C^2 domain of $\mathbb{H}^n$. Our result is the following:

Theorem 1.2. *Let $\Omega\subset\mathbb{H}^n$ be a C^2 bounded domain and $u\in C^3(\Omega)\cap C^2(\overline{\Omega})$ be a solution to* (1.5) *with $k\in\{1,\cdots,n\}$ and c_0 a positive constant. Then Ω is a geodesic ball B_R, and u is radially symmetric.*

By maximum principle, $u<0$ in Ω, and the solution to Dirichlet problem of $\sigma_k(D^2u-uI)=C_n^k$ is unique. In Theorem 1.2, if we assume the center of B_R is the origin, then $u(r)=\frac{\cosh r}{\cosh R}-1$ is the unique solution to (1.5), where r is the distance from 0, R and c_0 satisfy the relationship $\frac{\sinh R}{\cosh R}=c_0$.