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A De Giorgi Type Result to Divergence Degenerate Elliptic Equation with Bounded Coefficients Related to Hörmander's Vector Fields

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Abstract. In this paper, we consider the divergence degenerate elliptic equation with bounded coefficients constructed by Hörmander's vector fields. We prove a De Giorgi type result, i.e., the local Hölder continuity for the weak solutions to the equation by providing a De Giorgi type lemma and extending the Moser iteration to the setting here. As a consequence, the Harnack inequality of weak solutions is also given.

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1 Introduction

In 1957, De Girogi has found the local Hölder continuity of weak solutions to the following divergence elliptic equation with bounded coefficients

$$Lu=-\sum_{i,j=1}^{n} D_i\left(a^{ij}(x)D_ju\right)=0, \qquad x\in\mathbb{R}^n$$

and a priori estimate of Hölder norm (see [1]). Nash in [2] used a different approach and derived the similar result to the parabolic equation with bounded coefficients. In [3] Hou and Niu have taken into account of Nash's approach to obtain the Hölder regularity and Harnack inequality to divergence parabolic equation related to Hörmander's vector fields. Moser [4] developed a new method (nowadays it has been called the Moser iteration) and applied it to prove forenamed results with respect to elliptic and parabolic equations. These important ideas opened a new prospect for the study of regularity to

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partial differential equations. A natural and interesting problem is whether De Giorgi's result is true to the divergence degenerate elliptic equation related to Hörmander's vector fields. We will affirmatively answer it.

Hörmander introduced the square sum operator constructed by smooth vector fields and proved that it is hypoelliptic if vector fields satisfy the finite rank condition (see [5]). Many authors continued his research ([6–11]) and obtained numerous insight, such as, fundamental solutions ([12]), the Poincaré inequality ([13]), potential estimates ([14]) and sub-elliptic estimates ([15, 16]). Nagel, Stein and Wainger ([17]) deduced the basic properties of balls and metrics defined by Hörmander's vector fields, which are the starting point for treating many problems on the Hörmander square sum operator and related sub-elliptic operators. Lu found the Harnack inequality and Hölder continuousity for solutions to quasilinear degenerate elliptic equations formed by Hörmander's vector fields, see [18, 19].

Rothschild and Stein in [11] have proved regularity to the second order subelliptic equation. Xu and Zuily in [20] dealt with the interior regularity of weak solutions to the quasilinear degenerate elliptic system

$$\sum_{i,j=1}^{q} X_j^*\left(a^{ij}(x,u)X_i u^\alpha\right)=f^\alpha(x,u,Xu).$$

The Hölder regularity and Harnack inequality of the functions in the De Giorgi class related to Hörmander's vector fields are arrived at by Marchi in [21]. Bramanti and Brandolini in [22] gave regularity to the nondivergence degenerate elliptic equation of Hörmander's vector fields. The partial Hölder regularity for weak solutions to the quasilinear degenerate elliptic system was settled by Gao, Niu and Wang [23]. Dong and Niu in [24] obtained regularity of weak solutions to the nondiagonal quasilinear degenerate elliptic system

$$-X_\alpha^*\left(a_{ij}^{\alpha\beta}(x,u)X_\beta u^j\right)=g_i(x,u,Xu)-X_\alpha^* f_i^\alpha(x,u,Xu).$$

Schauder estimates to degenerate elliptic operators related to noncommutative vector fields have been derived in [25, 26] etc.

Throughout this paper we are concerned with the following divergence degenerate elliptic equation with bounded coefficients:

$$X_j^*\left(a^{ij}(x)X_i u\right)+b_i(x)X_i u+c(x)u=f(x)-X_i^* f^i(x) \qquad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $X_i=\sum_{k=1}^n b_{ik}(x)\frac{\partial}{\partial x_k}(b_{ik}(x)\in C^\infty(\Omega),\ i=1,\dots,q,\ q\le n)$ are smooth vector fields satisfying the finite rank condition, and the summation symbols in (1.1) are omitted. We assume that there exists $\Lambda>0$, such that

$$\Lambda^{-1}|\xi|^2\leqslant a^{ij}(x)\xi_i\xi_j\leqslant\Lambda|\xi|^2, \qquad \text{for } x\in\Omega,\ \xi\in\mathbb{R}^q, \tag{1.2}$$