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Positive Ground State Solutions for Schrödinger-Poisson System with General Nonlinearity and Critical Exponent

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Abstract. In this paper, we consider the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u+\eta\phi u=f(x,u)+u^5, & x\in\Omega, \\ -\Delta\phi=u^2, & x\in\Omega, \\ u=\phi=0, & x\in\partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in R^3, $\eta=\pm1$ and the continuous function f satisfies some suitable conditions. Based on the Mountain pass theorem, we prove the existence of positive ground state solutions.

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Key Words: Schrödinger-Poisson system; Sobolev critical exponent; positive ground state solution; Mountain pass theorem.

1 Introduction

In this paper, we study the following Schrödinger-Poisson system with general nonlinearity and critical exponent on bounded domain

$$\begin{cases} -\Delta u+\eta\phi u=f(x,u)+u^5, & x\in\Omega, \\ -\Delta\phi=u^2, & x\in\Omega, \\ u=\phi=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a smooth bounded domain in R^3, $\eta=\pm 1$, and the continuous function f satisfies some suitable conditions.

The Schrödinger equation, which is the first equation in system (1.1), describes quantum particles interacting with the electromagnetic field generated by the motion. An interesting class of Schrödinger equation is the case where the potential $\phi(x)$ is determined by the charge of the wave function itself, that is, when the second equation in system (1.1) holds, see [1]. From [2-5] and the references therein, we can learn more information about the physical relevance of the Schrödinger-Poisson system.

To the best of our knowledge, researchers only obtained a few results about the Schrödinger-Poisson system with critical exponent on bounded domain, see for instance [1], [6-11].

In [6], assuming that $\eta=\lambda$, $f(x,u)=\lambda u^{q-1}$, $\lambda>0$ and $1<q<2$, via using the variational method, the authors proved that system (1.1) has at least two positive solutions and one of the solutions is a ground state solution for all $\lambda\in(0,\lambda_*)$, where λ_* is a positive constant. In [7], let $\eta=-1$, $f(x,u)=\lambda f_\lambda(x)u^{q-1}$, $f_\lambda=\lambda f^+ + f^-$, $\lambda>0$ and $1<q<2$, by using the variational method and analytic techniques, they got that system (1.1) has at least two positive solutions and one of the solutions is a ground state solution for all $\lambda\in(0,\lambda_*)$, where λ_* is a positive constant. In [8], when $\eta=-1$, $f(x,u)=\lambda u^{q-1}$, $\lambda>0$ and $2<q<6$, by the Mountain pass theorem and the concentration compactness principle, they obtained that if $2<q\leq 4$, system (1.1) has at least one positive ground state solution for all $\lambda>\lambda_*$, where λ_* is a positive constant; if $4<q<6$, system (1.1) has at least one positive ground state solution for all $\lambda>0$.

In [9], let $\eta=\lambda$, $f(x,u)=\frac{\lambda}{u^r}$, $\lambda>0$ and $0<r<1$, the author got that system (1.1) has at least two positive solutions and one of the solutions is a ground state solution for all $\lambda\in(0,\lambda_*)$, where λ_* is a positive constant. In [10], assuming that $\eta=-1$, $f(x,u)=\frac{\lambda}{u^r}$, $\lambda>0$ and $0<r<1$, the authors proved that system (1.1) has at least two positive solutions for all $\lambda\in(0,\lambda_*)$, where λ_* is a positive constant. In [11], let $\eta=1$, $f(x,u)=\frac{\lambda}{|x|^\beta u^r}$, $\lambda>0$, $0<r<1$ and $0\leq\beta<\frac{5+r}{2}$, combining with the variational method and Nehari manifold method, two positive solutions of system (1.1) are obtained.

In [1], assuming that $\eta=1$, the nonlinear term $f:\Omega\times R\to R$ and its primitive F satisfies the following conditions:

(f_1) $|f(x,s)|\leq C(1+|s|^{p-1})$ for some $p\in(2,6)$, where C is positive constant;

(f_2) $f(x,s)=o(|s|)$ uniformly in x as $|s|\to 0$;

(f_3) $s\mapsto\frac{f(x,s)}{s^3}$ is nondecreasing on $(-\infty,0)\cup(0,+\infty)$;

(f_4) $\frac{F(x,s)}{|s|^4}\to+\infty$ uniformly in x as $s\to+\infty$.

Via the variational methods, the authors got that system (1.1) has at least one nontrivial ground state solution.

On the basis of the above literature, especially [1], we continues to study system (1.1) with general nonlinearity and critical exponent on bounded domain. We assume that $\eta=\pm 1$ and the nonlinear term $f:\Omega\times R\to R$ satisfies the following assumptions