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Well-Posedness and Blow-Up for the Fractional Schrödinger-Choquard Equation

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Abstract. In this paper, we study the well-posedness and blow-up solutions for the fractional Schrödinger equation with a Hartree-type nonlinearity together with a power-type subcritical or critical perturbations. For nonradial initial data or radial initial data, we prove the local well-posedness for the defocusing and the focusing cases with subcritical or critical nonlinearity. We obtain the global well-posedness for the defocusing case, and for the focusing mass-subcritical case or mass-critical case with initial data small enough. We also investigate blow-up solutions for the focusing mass-critical problem.

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Key Words: Fractional Schrödinger equation; Hartree-type nonlinearity; well-posedness; blow-up.

1 Introduction

In this paper we consider the following Cauchy problem for the fractional nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u=(-\Delta)^{\alpha}u+\lambda\left(|u|^k u+\left(|x|^{-\gamma}*|u|^2\right)u\right), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^N, \\ u(0,x)=\varphi(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $N \geq 1$, $0 < \alpha < 1$, $0 < \gamma < N$, $0 \leq k \leq \frac{4\alpha}{N}$, $\lambda = \pm 1$, $*$ denotes the convolution in $\mathbb{R}^N$, i is the imaginary unit and $u = u(t,x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is the unknown complex-valued function. The fractional Laplace operator $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u = \frac{1}{(2\pi)^N} \int e^{ix\cdot\xi} |\xi|^{2\alpha} \mathcal{F}[u](\xi) d\xi = \mathcal{F}^{-1}\left[|\xi|^{2\alpha}\mathcal{F}[u](\xi)\right],20$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the Fourier inverse transform in $\mathbb{R}^N$, respectively. When $\lambda = 1$, (1.1) is referred to be defocusing fractional NLS, while $\lambda = -1$, (1.1) is referred to be focusing fractional NLS.

In recent years, there has been wide interest in applying fractional Laplacians to model physical phenomena. By extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, Laskin in [1,2] used the theory of functionals over functional measure generated by the Lévy stochastic process to deduce the following nonlinear fractional Schrödinger equation

$$i\partial_t u = (-\Delta)^\alpha u + f(u), \tag{1.2}$$

where $0 < \alpha < 1, f(u) = |u|^k u$. The parameter $0 < \alpha < 1$ is the corresponding index of the Lévy stable processes, see [1,2]. Eq. (1.2) with $\alpha = \frac{1}{2}$ has been also used as models to describe Boson stars. Recently, an optical realization of the fractional Schrödinger equation was proposed by Longhi [3]. For the nonlinearity $|u|^p u$, the well-posedness and ill-posedness in the Sobolev space H^α have been investigated in [4,5]. In [6], Boulenger, Himmelsbach and Lenzmann have obtained a general criterion for blow-up of radial solution of (1.1) with $k \geq \frac{4\alpha}{N}$ and $N \geq 2$. Although a general existence theorem for blow-up solutions of this problem is still an open problem, it has been strongly supported by numerical evidence [7].

Also, Eq. (1.2) has attracted more and more attention in both physics and mathematics, see [4–6,8–16]. For the Hartree-type nonlinearity

$$f(u) = \left(|x|^{-\gamma} * |u|^2\right) u,$$

Cho et al. in [8] proved existence and uniqueness of local and global solutions of (1.2). In [9] the authors showed the existence of blow-up solutions. The dynamical properties of blow-up solutions have been investigated in [10,11]. The stability and instability of standing waves have been studied in [12]. For other kinds of fractional Schrödinger equations in which the Hartree-type nonlinearity being replaced by a sublinearity, the orbital stability of standing waves has been studied in [13,14].

Recently, Bhattarai in [17] employed the concentration compactness techniques to prove existence and stability of standing waves for the following nonlinear fractional Schrödinger-Choquard equation

$$\begin{cases} i\partial_t u = (-\Delta)^\alpha u - |u|^k u - (|x|^{-\gamma} * |u|^p)|u|^{p-2}u, & (t,x) \in [0,T) \times \mathbb{R}^N, \\ u(0,x) = \varphi(x), \end{cases}$$