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Analytic Smoothing Effect of the Time Variable for the Spatially Homogeneous Landau Equation

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Abstract. In this work, we study the Cauchy problem of the spatially homogeneous Landau equation with hard potentials in a close-to-equilibrium framework. We prove that the solution to the Cauchy problem enjoys the analytic regularizing effect of the time variable with an L^2 initial datum for positive time. So that the smoothing effect of the Cauchy problem for the spatially homogeneous Landau equation with hard potentials is exactly same as heat equation.

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1 Introduction

In this work, we are concerned with the following Cauchy problem of spatially homogenous Landau equation

$$\begin{cases} \partial_t F = Q(F,F), \\ F|_{t=0} = F_0, \end{cases} \tag{1.1}$$

where $F = F(t,v) \geq 0$ is the density distribution function at time $t \geq 0$, with the velocity variable $v \in \mathbb{R}^3$. The Landau bilinear collision operator is defined by

$$Q(G,F)(v) = \sum_{i,j=1}^{3} \partial_i \left(\int_{\mathbb{R}^3} a_{ij}(v-v_*)[G(v_*)\partial_j F(v) - \partial_j G(v_*) F(v)] \mathrm{d}v_* \right),$$

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where

$$a_{ij}(v)=(\delta_{ij}|v|^2-v_iv_j)|v|^\gamma,\quad \gamma\geq -3,$$

is a symmetric non-negative matrix such that $a_{ij}(v)v_iv_j=0$. Here, γ is a parameter which leads to the classification of the hard potential if $\gamma>0$, Maxwellian molecules if $\gamma=0$, soft potential if $\gamma\in]-3,0[$ and Coulombian potential if $\gamma=-3$.

The Landau equation was introduced as a limit of the Boltzmann equation when the collisions become grazing in [1,2]. The global existence, and uniqueness of classical solutions for the spatially homogeneous Landau equation with hard potentials, regularizing effects, and large-time behavior have been addressed by Desvillettes and Villani [3,4]. Moreover, they proved the smoothness of the solution in $C^\infty(]0,\infty[;\mathcal{S}(\mathbb{R}^3))$. Carrapatoso [5] proved an exponential in-time convergence to the equilibrium. In [6], the authors proved the solution is analytic of v variables for any $t>0$ and the Gevrey regularity in [7,8].

Let μ be the Maxwellian distribution

$$\mu(v)=(2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}},$$

we shall linearize the Landau equation (1.1) around μ with the fluctuation of the density distribution function

$$F(t,v)=\mu(v)+\sqrt{\mu}(v)f(t,v),$$

since $Q(\mu,\mu)=0$, the Cauchy problem (1.1) for $f=f(t,v)$ takes the form

$$\begin{cases}\partial_t f+\mathcal{L}(f)=\Gamma(f,f),\\ f|_{t=0}=f_0,\end{cases}\tag{1.2}$$

with $F_0(v)=\mu+\sqrt{\mu}f_0(v)$, where

$$\begin{aligned}&\Gamma(g,h)=\mu^{\frac{-1}{2}}Q(\mu^{\frac{1}{2}}g,\mu^{\frac{1}{2}}h),\\ &\mathcal{L}(f)=\mathcal{L}_1f+\mathcal{L}_2f,\quad \mathcal{L}_1f=-\Gamma(\mu^{\frac{1}{2}},f),\quad \mathcal{L}_2f=-\Gamma(f,\mu^{\frac{1}{2}}).\end{aligned}$$

In the case of the Maxwellian molecules, Villani [4] has proved a linear functional inequality between entropy and entropy dissipation by constructive methods, from which one deduces an exponential convergence of the solution to the Maxwellian equilibrium in relative entropy, which in turn implies an exponential convergence in L^1-distance. In [9], Desvillettes and Villani have proved a functional inequality for entropy dissipation is not linear, from which one obtains a polynomial in time convergence of solutions towards the equilibrium in relative entropy, which implies the same type of convergence in L^1-distance. In [10], the authors studied the spatially homogeneous Landau equation and non-cutoff Boltzmann equation in a close-to-equilibrium framework and proved the Gelfand-Shilov smoothing effect (see also [11,12]). Guo [13] constructed global classical solutions for the spatially inhomogeneous Landau equation near a global Maxwellian in