

## Coupled Lamé System with a Viscoelastic Term and a Strong Discrete Time Delays

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**Abstract.** In this paper, we consider a coupled Lamé system with a viscoelastic damping in the first equation and two strong discrete time delays. We prove its existence by using the Faedo-Galerkin method and establish an exponential decay result by introducing a suitable Lyapunov functional.

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**Key Words:** Lamé system; coupled system; Lyapunov function; delay term.

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### 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^3$  with smooth boundary  $\partial\Omega$ . We consider the following a coupled Lamé system

$$\left\{ \begin{array}{ll} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \int_0^t \omega(s) \Delta u(t-s) ds \\ \quad - \lambda_1 \Delta u_t(x,t) - \mu_1 \Delta u_t(x,t - \tau_1) = 0, & \text{in } \Omega \times (0, +\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) - \lambda_2 \Delta v_t(x,t) - \mu_2 \Delta v_t(x,t - \tau_2) = 0, & \text{in } \Omega \times (0, +\infty), \\ u(x,t) = v(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)), & \text{in } \Omega, \\ u_t(x, t - \tau_1) = f_1(x, t - \tau_1), & \text{in } \Omega \times [0, \tau_1], \\ v_t(x, t - \tau_2) = f_2(x, t - \tau_2), & \text{in } \Omega \times [0, \tau_2], \end{array} \right. \quad (1.1)$$

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where  $\mu_1, \mu_2$  are positive constants,  $\tau_1, \tau_2 > 0$  are the delays, and  $(u_0, u_1, v_0, v_1)$  are given history and initial data. Here  $\Delta$  denotes the Laplacian operator and  $\Delta_e$  denotes the elasticity operator, which is the  $3 \times 3$  matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div} u), \quad u = (u_1, u_2, u_3)^T$$

and  $\mu$  and  $\lambda$  are the Lamé constants which satisfy the conditions

$$0 < \mu, \quad 0 \leq \lambda + \mu. \tag{1.2}$$

They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions.

We review some related results, R. Racke [1] considered the system

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t-\tau) + b\theta_x(x,t) = 0, & \text{in } (0,L) \times (0,\infty), \\ \theta_t(x,t) - d\theta_{xx}(x,t) + bu_{tx}(x,t) = 0, & \text{in } (0,L) \times (0,\infty). \end{cases}$$

The author showed the well-posedness and stability of the system without delay. In [2] the authors examined a transmission problem with a viscoelastic term and a delay

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \int_0^t g(t-s)u_{xx}(x,s)ds + \mu_1 u_t(x,t) \\ \quad + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in (L_1, L_2) \times (0,+\infty). \end{cases}$$

Under appropriate hypotheses on the relaxation function and the relationship between the weight of the damping and the weight of the delay, they proved the well-posedness result and exponential decay of the energy. In [3], M. I. Mustafa considered the following system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + f_1(u,v) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) - \Delta v(x,t) + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + f_2(u,v) = 0, & \text{in } \Omega \times (0,+\infty), \\ u = v = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ (u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1, & \text{in } \Omega. \end{cases}$$

The author proved the well-posedness and, for a wider class of relaxation functions, established a generalized stability results for this system. For Timoshenko-type systems, A. Guesmia and et al. [4] considered the following system

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - k_1(\varphi_x + \psi)_x(x,t) + \lambda_1 \varphi_t(x,t) + \mu_1 \varphi_t(x,t - \tau_1) = 0, \\ \rho_2 \psi_{tt}(x,t) - k_2 \psi_{xx}(x,t) + k_1(\varphi_x + \psi)(x,t) + \lambda_2 \psi_t(x,t) + \mu_2 \psi_t(x,t - \tau_2) = 0, \end{cases} \tag{1.3}$$