On the Well-Posedness Problem of the Anisotropic Porous Medium Equation with a Variable Diffusion Coefficient

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Abstract. The initial-boundary value problem of an anisotropic porous medium equation

\[ u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( a(x,t) |u|^\alpha_i u_{x_i} \right) \]

is studied. Compared with the usual porous medium equation, there are two different characteristics in this equation. One lies in its anisotropic property, another one is that there is a nonnegative variable diffusion coefficient \( a(x,t) \) additionally. Since \( a(x,t) \) may be degenerate on the parabolic boundary \( \partial \Omega \times (0,T) \), instead of the boundedness of the gradient \( |\nabla u| \) for the usual porous medium, we can only show that \( \nabla u \in L^\infty(0,T;L^2_{\text{loc}}(\Omega)) \). Based on this property, the partial boundary value conditions matching up with the anisotropic porous medium equation are discovered and two stability theorems of weak solutions can be proved naturally.

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Key Words: Anisotropic porous medium equation; variable diffusion coefficient; stability; partial boundary condition.

1 Introduction

Let \( \Omega \subset \mathbb{R}^N \) be a bounded smooth domain, \( T \in (0,\infty) \) be a given positive constant. The well-posedness problem and the regularity of weak solutions to the usual porous medium equation

\[ u_t = \Delta u^m, \quad (x,t) \in Q_T = \Omega \times (0,T), \]

\[ \text{for } m > 1 \]

is studied. Compared with the usual porous medium equation, there are two different characteristics in this equation. One lies in its anisotropic property, another one is that there is a nonnegative variable diffusion coefficient \( a(x,t) \) additionally. Since \( a(x,t) \) may be degenerate on the parabolic boundary \( \partial \Omega \times (0,T) \), instead of the boundedness of the gradient \( |\nabla u| \) for the usual porous medium, we can only show that \( \nabla u \in L^\infty(0,T;L^2_{\text{loc}}(\Omega)) \). Based on this property, the partial boundary value conditions matching up with the anisotropic porous medium equation are discovered and two stability theorems of weak solutions can be proved naturally.

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or its revised form

\[ \beta(u)_t = \Delta u, \quad \beta(u) = |u|^{\frac{1}{m}} \text{sign} u, \quad (x,t) \in Q_T, \tag{1.2} \]

were addressed from the sixties to the eighties in the last century, one can refer to [1–6] and the references therein. Meanwhile, by a parabolic version of De Giorgi’s technique, DiBenedetto [7] studied the regularity of a general equation

\[ \beta(u)_t = \nabla \cdot \vec{a}(x,t,u,\nabla u) + b(x,t,u,\nabla u), \quad (x,t) \in Q_T, \tag{1.3} \]

with suitable assumptions on \( \vec{a} \) and \( b \). Later, Ziemer [8] approached a similar problem with Moser’s iteration technique.

The main problems arising in the study of complicated real physical processes are related primarily, to the nonlinearity of Eqs. (1.1)-(1.3). The first consequence of nonlinearity is the absence of a superposition principle, which applies to linear homogeneous problems. This leads to an inexhaustible set of possible directions of evolution of a dissipative process and also determines the appearance in a continuous medium of discrete spatiotemporal scales. These characterize the properties of the nonlinear medium, which are independent of external factors. More precisely, Eq. (1.1) exhibits the ideal barotropic gas through a porous medium, while Eq. (1.3) is suggested as a model to describe the spread of epidemic disease in heterogeneous environments or it is used as the mathematical description for the dynamics of fluids with different conductivities in different directions. An isotropic example of Eq. (1.3) is

\[ u_t = \text{div}(a(x) \nabla u^m) + \sum_{i=1}^{N} \frac{\partial b_i(u^m)}{\partial x_i}, \quad (x,t) \in Q_T. \tag{1.4} \]

The well-posedness problem of this equation was first studied in [9] by the author. We found that, if one wants to prove the uniqueness (or the stability) of weak solutions, the initial value condition

\[ u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.5} \]

is always needed, but the homogeneous boundary value condition

\[ u(x,t) = 0, \quad (x,t) \in \partial \Omega \times (0,T), \tag{1.6} \]

may be dispensable. The anisotropic evolutionary parabolic equation also has provoked people’s attention. Song [10, 11] studied the existence and the uniqueness of the “very weak” solution of the anisotropic porous medium equation modeled by

\[ u_t = \sum_{i=1}^{N} (u^{m_i})_{x_i}, \quad (x,t) \in Q_T. \tag{1.7} \]

Henriques [12] established an interior regularity result for the solutions of (1.7). Li [13] developed the finite element method to derive a special analytical solution of Eq. (1.7) for time-independent diffusion.