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On the Well-Posedness Problem of the Anisotropic Porous Medium Equation with a Variable Diffusion Coefficient

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Abstract. The initial-boundary value problem of an anisotropic porous medium equation

$$u_t=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a(x,t)|u|^{\alpha_i}u_{x_i}\right)+\sum_{i=1}^{N}\frac{\partial f_i(u,x,t)}{\partial x_i}$$

is studied. Compared with the usual porous medium equation, there are two different characteristics in this equation. One lies in its anisotropic property, another one is that there is a nonnegative variable diffusion coefficient $a(x,t)$ additionally. Since $a(x,t)$ may be degenerate on the parabolic boundary $\partial\Omega\times(0,T)$, instead of the boundedness of the gradient $|\nabla u|$ for the usual porous medium, we can only show that $\nabla u\in L^\infty(0,T;L^2_{\rm loc}(\Omega))$. Based on this property, the partial boundary value conditions matching up with the anisotropic porous medium equation are discovered and two stability theorems of weak solutions can be proved naturally.

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1 Introduction

Let $\Omega\subset\mathbb{R}^N$ be a bounded smooth domain, $T\in(0,\infty)$ be a given positive constant. The well-posedness problem and the regularity of weak solutions to the usual porous medium equation

$$u_t=\Delta u^m,\quad (x,t)\in Q_T=\Omega\times(0,T), \tag{1.1}$$

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or its revised form

$$\beta(u)_t=\Delta u,\ \beta(u)=|u|^{\frac{1}{m}}\mathrm{sign}u, \quad (x,t)\in Q_T, \tag{1.2}$$

were addressed from the sixties to the eighties in the last century, one can refer to [1–6] and the references therein. Meanwhile, by a parabolic version of De Giorgi's technique, DiBenedetto [7] studied the regularity of a general equation

$$\beta(u)_t=\nabla\cdot\vec{a}(x,t,u,\nabla u)+b(x,t,u,\nabla u), \quad (x,t)\in Q_T, \tag{1.3}$$

with suitable assumptions on $\vec{a}$ and b. Later, Ziemer [8] approached a similar problem with Moser's iteration technique.

The main problems arising in the study of complicated real physical processes are related primarily, to the nonlinearity of Eqs. (1.1)-(1.3). The first consequence of nonlinearity is the absence of a superposition principle, which applies to linear homogeneous problems. This leads to an inexhaustible set of possible directions of evolution of a dissipative process and also determines the appearance in a continuous medium of discrete spatiotemporal scales. These characterize the properties of the nonlinear medium, which are independent of external factors. More precisely, Eq. (1.1) exhibits the ideal barotropic gas through a porous medium, while Eq. (1.3) is suggested as a model to describe the spread of epidemic disease in heterogeneous environments or it is used as the mathematical description for the dynamics of fluids with different conductivities in different directions. A isotropic example of Eq. (1.3) is

$$u_t=\mathrm{div}(a(x)\nabla u^m)+\sum_{i=1}^N\frac{\partial b_i(u^m)}{\partial x_i}, \quad (x,t)\in Q_T. \tag{1.4}$$

The well-posedness problem of this equation was first studied in [9] by the author. We found that, if one wants to prove the uniqueness (or the stability) of weak solutions, the initial value condition

$$u(x,0)=u_0(x), \quad x\in\Omega, \tag{1.5}$$

is always needed, but the homogeneous boundary value condition

$$u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T), \tag{1.6}$$

may be dispensable. The anisotropic evolutionary parabolic equation also has provoked people's attention. Song [10, 11] studied the existence and the uniqueness of the "very weak" solution of the anisotropic porous medium equation modeled by

$$u_t=\sum_{i=1}^N(u^{m_i})_{x_i}, \quad (x,t)\in Q_T. \tag{1.7}$$

Henriques [12] established an interior regularity result for the solutions of (1.7). Li [13] developed the finite element method to derive a special analytical solution of Eq. (1.7) for time-independent diffusion.