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A Logarithmic Decay of the Energy for the Hyperbolic Equation with Supercritical Damping

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Abstract. We are concerned with the following quasilinear wave equation involving variable sources and supercritical damping:

$$u_{tt}-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)+|u_t|^{m-2}u_t=|u|^{q(x)-2}u.$$

Generally speaking, when one tries to use the classical multiplier method to analyze the asymptotic behavior of solutions, an inevitable step is to deal with the integral $\int_\Omega |u_t|^{m-2}u_t u\mathrm{d}x$. A usual technique is to apply Young's inequality and Sobolev embedding inequality to use the energy function and its derivative to control this integral for the subcritical or critical damping. However, for the supercritical case, the failure of the Sobolev embedding inequality makes the classical method be impossible. To do this, our strategy is to prove the rate of the integral $\int_\Omega |u|^m \mathrm{d}x$ grows polynomially as a positive power of time variable t and apply the modified multiplier method to obtain the energy functional decays logarithmically. These results improve and extend our previous work [12]. Finally, some numerical examples are also given to authenticate our results.

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1 Introduction

Let $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ be a bounded domain and $\partial\Omega$ be Lipschitz continuous. Throughout

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this paper, we investigate the following quasilinear hyperbolic problem with variable sources

$$\begin{cases} u_{tt}-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+|u_t|^{m-2}u_t=|u|^{q(x)-2}u, & (x,t)\in\Omega\times(0,T)=:Q_T, \\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T)=:\Gamma_T, \\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x), & x\in\Omega. \end{cases} \tag{1.1}$$

In what follows, we always assume that the exponent m is a constant, and the exponents $p(x)$,$q(x)$ are continuous functions on $\overline{\Omega}$ with the logarithmic module of continuity satisfying:

$$2\le p_1\le p(x)\le p_2<\infty,\ \ 1\le m<\infty,\ \ 1\le q_1\le q(x)\le q_2<\infty, \tag{1.2a}$$

$$\forall x,y\in\Omega,\ \ |x-y|<1,\ \ |p(x)-p(y)|+|q(x)-q(y)|\le\omega(|x-y|), \tag{1.2b}$$

where

$$\lim_{\tau\to 0^+}\sup\omega(\tau)\ln\frac{1}{\tau}=C<\infty. \tag{1.3}$$

Equations with variable exponents of nonlinearities are usually treated as equations with nonstandard growth conditions. The problem with nonstandard growth conditions occurs in many mathematical models of applied science, such as viscoelastic fluids, electro-rheological fluids, processes of filtration through a porous media, fluids with temperature-dependent viscosity, the image processing etc, refer to [1–3] and the references therein. For the presence of strong damping, Antontsev [4, 5] and Guo [6, 7] applied energy estimate methods and differential inequality argument to discuss finite-time blow-up solutions to the corresponding problem in the case of negative and positive initial energy, respectively. Later, Messaoudi and Talahmeh [8,9] applied similar techniques mentioned above to study blow-up properties of solutions to Problem (1.1). Afterward, Liao [10] investigated the upper and lower bounds of the blow-up time and energy decay for a viscoelastic wave equation with variable exponents. Besides, Liao and Tan [11] took the method into dealing with a Petrovsky equation with damping and variable-exponent sources.

Therefore, another engineering concern is to reduce the vibrations/oscillations of the structure. From the mathematical point of view, this means stability analysis and, more generally, qualitative analysis of the long-time behavior of the corresponding solutions. However, it is well known that the source term causes finite-time blow-up of solutions and drives the equation to possible instability while the damping term prevents finite-time blow-up of the solution and drives the equation toward stability. So, it is of interest to explore the mechanism of how the sources dominate the dissipation. As far as we know, such a result is seldom found. In [12], the authors and Liao obtained the following result in the case of subcritical damping:

Theorem 1.1. ([12]) *If the following conditions are fulfilled*