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A Note on a Multi-Dimensional Radiating Gas Model with Nonlinear Radiative Inhomogeneity

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Abstract. In this paper, we consider the Cauchy problem of a multi-dimensional radiating gas model with nonlinear radiative inhomogeneity. Such a model gives a good approximation to the radiative Euler equations, which are a fundamental system in radiative hydrodynamics with many practical applications in astrophysical and nuclear phenomena. One of our main motivations is to attempt to explore how nonlinear radiative inhomogeneity influences the behavior of entropy solutions. Simple but different phenomena are observed on relaxation limits. On one hand, the same relaxation limit such as the hyperbolic-hyperbolic type limit is obtained, even for different scaling. On the other hand, different relaxation limits including hyperbolic-hyperbolic type and hyperbolic-parabolic type limits are obtained, even for the same scaling if different conditions are imposed on nonlinear radiative inhomogeneity.

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Key Words: Radiating gas model; nonlinear radiative inhomogeneity; entropy solution; global well-posedness; relaxation limit.

1 Introduction

In this paper, we consider a multi-dimensional radiating gas model with nonlinear radiative inhomogeneity

$$\begin{cases} \partial_t u + \mathrm{div} f(u) + \mathrm{div} q = 0, \\ -\nabla \mathrm{div} q + q + \nabla g(u) = 0. \end{cases} \tag{1.1}$$

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Here both $u:=u(x,t):\mathbb{R}^d\times(0,\infty)\to\mathbb{R}$ and $q:=q(x,t):\mathbb{R}^d\times(0,\infty)\to\mathbb{R}^d$ are unknown. The flux function $f(u)=(f_1(u),\cdots,f_d(u)):\mathbb{R}\to\mathbb{R}^d$ and the nonlinear radiative inhomogeneity $g(u):\mathbb{R}\to\mathbb{R}$ are given such as the particular example

$$g(u)=u|u|^{m-1}\quad\text{for some}\;\; m\geq 1. \tag{1.2}$$

For convenience, the system (1.1) is equivalent to the scalar balance law

$$\partial_t u+\mathrm{div} f(u)=-g(u)+K*g(u). \tag{1.3}$$

Indeed, one can solve the $(1.1)_2$ to obtain div$q$ by u and substitute it into $(1.1)_1$. Then the convolution model (1.3) is derived. Here $K(x)$ is the fundamental solution to the elliptic operator $-\Delta+I$ and "$*$" denotes the convolution with respect to the space variable x. The nonlocal force in (1.3) reflects the global influence of heat sources or gravitation fields and appears in radiative hydrodynamics [1]. A typical choice is $g(u)=\sigma u^4$ with $\sigma\geq 0$ which derives from Planck's law of black body radiation.

We are concerned with the Cauchy problem of Eq. (1.3) with initial data

$$u(x,0)=u_0(x),\quad x\in\mathbb{R}^d. \tag{1.4}$$

One of our main motivations in this paper is to attempt to extend the case of linear radiative inhomogeneity in [2] to the nonlinear radiative inhomogeneity and explore how nonlinear radiative inhomogeneity influences the behavior of entropy solution, which is defined as follow:

Definition 1.1 (entropy solution). *A bounded measurable function $u:\mathbb{R}^d\times(0,+\infty)\to\mathbb{R}$ is said to be an entropy solution to Eq.* (1.3) *with initial datum $u_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ if it verifies the inequality*

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}[\eta(u)\partial_t\psi+\Psi(u)\cdot\nabla\psi]\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^d}\eta(u_0(x))\psi(x,0)\mathrm{d}x\\ \geq&\int_0^T\int_{\mathbb{R}^d}\eta'(u)[g(u)-K*g(u)]\psi\mathrm{d}x\mathrm{d}t\end{aligned} \tag{1.5}$$

for any convex entropy η with flux $\Psi=(\Psi_1,\cdots,\Psi_d):\mathbb{R}\to\mathbb{R}^d$ given by

$$\Psi_\alpha(u)=\int^u f'_\alpha(s)\eta'(s)\mathrm{d}s, \tag{1.6}$$

where $\alpha=1,\cdots,d$, and for any nonnegative Lipschitz continuous test function ψ with compact support on $\mathbb{R}^d\times[0,T)$.

Now our main theorems are stated as follows.