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Life-Spans and Blow-Up Rates for a p-Laplacian Parabolic Equation with General Source

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Abstract. This article investigates the blow-up results for the initial boundary value problem to the quasi-linear parabolic equation with p-Laplacian

$$u_t-\nabla\cdot\Big(|\nabla u|^{p-2}\nabla u\Big)=f(u),$$

where $p\geq 2$ and the function $f(u)$ satisfies

$$\alpha\int_0^u f(s)\mathrm{d}s\leq uf(u)+\beta u^p+\gamma,\quad u>0$$

for some positive constants α,β,γ with $0<\beta\leq\frac{(\alpha-p)\lambda_{1,p}}{p}$, which has been studied under the initial condition $J_p(u_0)<0$. This paper generalizes the above results on the following aspects: a new blow-up condition is given, which holds for all $p>2$; a new blow-up condition is given, which holds for $p=2$; some new lifespans and upper blow-up rates are given under certain conditions.

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Key Words: p-Laplacian; parabolic equation; blow-ups; life-spans; blow-up rates.

1 Introduction

This article is concerned with the blow-up results on the initial boundary value problem

$$\begin{cases}u_t(x,t)-\nabla\cdot\big(|\nabla u(x,t)|^{p-2}\nabla u(x,t)\big)=f(u(x,t)), & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x)\geq 0, & x\in\overline{\Omega},\end{cases}\tag{1.1}$$

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where $p\geq 2$, Ω is a bounded domain in $\mathbb{R}^n$ $(n\geq 1)$ with sufficiently smooth boundary $\partial\Omega$ and the function $f(u)$ is locally Lipschitz continuous on $\mathbb{R}$, $f(0)=0$ and $f(u)>0$ if $u>0$, which satisfies

$$\alpha F(u):=\alpha\int_0^u f(s)\mathrm{d}s\leq uf(u)+\beta u^p+\gamma,\quad u>0 \tag{C_p}$$

for some positive constants α,β,γ with

$$0<\beta\leq\frac{(\alpha-p)\lambda_{1,p}}{p}, \tag{1.2}$$

throughout this article $\lambda_{1,p}$ is the principal eigenvalue of the p-Laplacian Δ_p. For the function $f(u)$ satisfying other conditions, there are also many excellent research results on the blow-up issues of the problem (1.1), such as [1–3].

The blow-up issues for the problem (1.1) has been studied by Chung and Choi [4], the details are outlined below:

(RES1) If $f(u)$ satisfies (C_p) and if the initial value $u_0\in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ as $p>2$ or $u_0\in C^1(\overline{\Omega})$ as $p=2$ satisfies $J_p(0)<0$, then the solution for the problem (1.1) blows up at some finite time T, which is given by

$$0<T\leq T_{U1}:=\frac{\left(1+\sqrt{\frac{\alpha}{2}}\right)^2}{(\alpha-2)^2}\frac{\int_\Omega|u_0|^2\mathrm{d}x}{-J_p(0)},$$

where

$$J_p(t)=J_p(u(t))=\int_\Omega\left(\frac{1}{p}|\nabla u|^p-(F(u)-\gamma)\right)\mathrm{d}x. \tag{1.3}$$

From [5, the blow-up criterion 4], it is well known that if $f(u)$ satisfies the condition

$$(2+\varepsilon)\int_0^u f(s)\mathrm{d}s\leq uf(u)+\gamma,\quad u>0,\ \varepsilon>0, \tag{B}$$

then the blow-up criterion for the problem (1.1) with $p=2$ is given by

$$J_2(0)<\frac{(\alpha-1)\gamma|\Omega|}{\alpha}, \tag{1.4}$$

which implies that $J_2(0)\geq 0$ or $J_2(0)<0$. Therefore, from **(RES1)**, it is well known that

1. the blow-up condition is not given for possible $J_p(u_0)\geq 0$;
2. the life-spans is still unresolved when for possible $J_p(u_0)\geq 0$;
3. the blow-up rate is still unresolved for both $J_p(u_0)<0$ and possible $J_p(u_0)\geq 0$.