Life-Spans and Blow-Up Rates for a $p$-Laplacian Parabolic Equation with General Source

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Abstract. This article investigates the blow-up results for the initial boundary value problem to the quasi-linear parabolic equation with $p$-Laplacian

$$u_t - \nabla \cdot \left( |\nabla u|^{p-2} \nabla u \right) = f(u),$$

where $p \geq 2$ and the function $f(u)$ satisfies

$$a \int_0^u f(s) \, ds \leq uf(u) + \beta u^p + \gamma, \quad u > 0$$

for some positive constants $a, \beta, \gamma$ with $0 < \beta \leq \frac{(a-p)\lambda_1 p}{p}$, which has been studied under the initial condition $J_p(u_0) < 0$. This paper generalizes the above results on the following aspects: a new blow-up condition is given, which holds for all $p > 2$; a new blow-up condition is given, which holds for $p = 2$; some new lifespans and upper blow-up rates are given under certain conditions.

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1 Introduction

This article is concerned with the blow-up results on the initial boundary value problem

$$\begin{cases}
  u_t(x,t) - \nabla \cdot (|\nabla u(x,t)|^{p-2} \nabla u(x,t)) = f(u(x,t)), & (x,t) \in \Omega \times (0,T), \\
  u(x,t) = 0, & (x,t) \in \partial \Omega \times (0,T), \\
  u(x,0) = u_0(x) \geq 0, & x \in \Omega,
\end{cases}$$

(1.1)

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where $p \geq 2$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with sufficiently smooth boundary $\partial \Omega$ and the function $f(u)$ is locally Lipschitz continuous on $\mathbb{R}$, $f(0) = 0$ and $f(u) > 0$ if $u > 0$, which satisfies
\[
\alpha F(u) := \alpha \int_0^u f(s)ds \leq uf(u) + \beta u^p + \gamma, \quad u > 0
\] (C_p)
for some positive constants $\alpha, \beta, \gamma$ with
\[
0 < \beta \leq \frac{(\alpha - p)\lambda_{1,p}}{p}, \quad (1.2)
\]
throughout this article $\lambda_{1,p}$ is the principal eigenvalue of the $p$-Laplacian $\Delta_p$. For the function $f(u)$ satisfying other conditions, there are also many excellent research results on the blow-up issues of the problem (1.1), such as [1–3].

The blow-up issues for the problem (1.1) has been studied by Chung and Choi [4], the details are outlined below:

**RES1** If $f(u)$ satisfies (C_p) and if the initial value $u_0 \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega)$ as $p > 2$ or $u_0 \in C^1(\overline{\Omega})$ as $p = 2$ satisfies $J_p(0) < 0$, then the solution for the problem (1.1) blows up at some finite time $T$, which is given by
\[
0 < T \leq T_{U_1} := \frac{(1 + \sqrt{2})^2}{(\alpha - 2)^2} \frac{\int_{\Omega} |u_0|^2 \, dx}{-J_p(0)},
\]
where
\[
J_p(t) = J_p(u(t)) = \int_{\Omega} \left( \frac{1}{p} |\nabla u|^p - (F(u) - \gamma) \right) \, dx. \quad (1.3)
\]

From [5, the blow-up criterion 4], it is well known that if $f(u)$ satisfies the condition
\[
(2 + \epsilon) \int_0^u f(s)ds \leq uf(u) + \gamma, \quad u > 0, \ \epsilon > 0,
\] (B)
then the blow-up criterion for the problem (1.1) with $p = 2$ is given by
\[
J_2(0) \leq \frac{(\alpha - 1)\gamma |\Omega|}{\alpha}, \quad (1.4)
\]
which implies that $J_2(0) \geq 0$ or $J_2(0) < 0$. Therefore, from **RES1**, it is well known that

1. the blow-up condition is not given for possible $J_p(u_0) \geq 0$;
2. the life-spans is still unresolved when for possible $J_p(u_0) \geq 0$;
3. the blow-up rate is still unresolved for both $J_p(u_0) < 0$ and possible $J_p(u_0) \geq 0$. 
