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$W^{m,p(t,x)}$-Estimate for a Class of Higher-Order Parabolic Equations with Partially BMO Coefficients

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Abstract. We prove a global estimate in the Sobolev spaces with variable exponents to the solution of a class of higher-order divergence parabolic equations with measurable coefficients over the non-smooth domains. Here, it is mainly assumed that the coefficients are allowed to be merely measurable in one of the spatial variables and have a small BMO quasi-norm in the other variables at a sufficiently small scale, while the boundary of the underlying domain belongs to the so-called Reifenberg flatness. This is a natural outgrowth of Dong-Kim-Zhang's papers [1, 2] from the $W^{m,p}$-regularity to the $W^{m,p(t,x)}$-regularity for such higher-order parabolic equations with merely measurable coefficients with Reifenberg flat domain which is beyond the Lipschitz domain with small Lipschitz constant.

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1 Introduction

We devote this paper to a global estimate in the Sobolev spaces with a variable exponent for the solution of a higher-order divergence form the parabolic equation under weaker regularity assumptions on the coefficients, the variable exponent functions, and the underlying domains. Let Ω be a bounded domain of $\mathbb{R}^d$ for $d\geq 2$ with a rough boundary $\partial\Omega$ specified later. We set that $\Omega_T=(0,T)\times\Omega$ for $0<T<+\infty$ is a typical parabolic cylindrical domain in $\mathbb{R}\times\mathbb{R}^d$, and $\partial\Omega_T=\big((0,T)\times\partial\Omega\big)\cup\big(\{t=0\}\times\Omega\big)$ is its parabolic boundary of

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Ω_T. We write $D^\alpha u = D_1^{\alpha_1}\cdots D_d^{\alpha_d} u$ and $|\alpha| = \sum_{i=1}^d \alpha_i$ for multi-index $\alpha = (\alpha_1,\cdots,\alpha_d)$ with non-negative integer components. In this article, we consider the following Cauchy-Dirichlet problem of a $2m$-th order parabolic equation of divergence form with integer $m \geq 1$:

$$\begin{cases} \mathcal{P}_0 u := u_t + (-1)^m \sum\limits_{|\alpha|=m,|\beta|=m} D^\alpha \left(A^{\alpha\beta}(t,x) D^\beta u \right) = \sum\limits_{|\alpha|=m} D^\alpha f_\alpha, & \text{in } \Omega_T, \\ \sum\limits_{|\gamma|\leq m-1} |D^\gamma u| = 0, & \text{on } \partial\Omega_T, \end{cases} \tag{1.1}$$

where

$$\mathbf{f} = \left\{ f_\alpha : \Omega_T \to \mathbb{R}^d \,|\, |\alpha| = m \right\}$$

is the given tensorial-valued function of m-order in $L^2(\Omega_T)$ with regularity assumption specified later.

In the context, the solution for Problem (1.1) is understood in the following weak sense: we say that

$$u \in C^0\left((0,T); L^2(\Omega)\right) \cap L^2\left((0,T); H_0^m(\Omega)\right) \tag{1.2}$$

is a weak solution of (1.1) if there holds

$$\begin{aligned} &\int_{\Omega_T} u\varphi_t \mathrm{d}x\mathrm{d}t - \sum_{|\alpha|=m,|\beta|=m} \int_{\Omega_T} A^{\alpha\beta}(t,x) D^\beta u D^\alpha \varphi \mathrm{d}x\mathrm{d}t \\ =&(-1)^{m+1} \sum_{|\alpha|=m} \int_{\Omega_T} f_\alpha D^\alpha \varphi \mathrm{d}x\mathrm{d}t \end{aligned} \tag{1.3}$$

for any $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi = 0$ at $t = T$. Here, the coefficients $A^{\alpha\beta}$ for $|\alpha| = |\beta| = m$ are supposed to be uniformly ellipticity and boundedness, namely there exist two positive constants $0 < \mu \leq \Lambda$ such that

$$\sum_{|\alpha|=|\beta|=m} A^{\alpha\beta}(t,x)\xi_\alpha\xi_\beta \geq \mu|\xi|^2, \tag{1.4}$$

$$\sum_{|\alpha|=|\beta|=m} |A^{\alpha\beta}(t,x)| \leq \Lambda \tag{1.5}$$

for any $\xi = \{\xi_\alpha \in \mathbb{R}^d : |\alpha| = m\} \in \mathbb{R}^{md}$ and almost every $(t,x) \in \Omega_T$. We observe that for $\mathbf{f} \in L^2(\Omega_T)$, the Lax-Milgram theorem leads to that there exists a unique weak solution for the Cauchy-Dirichlet problem (1.1) with the standard $L^2(\Omega_T)$-estimate

$$\|D^m u\|_{L^2(\Omega_T)} \leq c\|\mathbf{f}\|_{L^2(\Omega_T)}, \tag{1.6}$$

where $c = c(\mu,\Lambda,d,m)$ is a positive constant, see [3]. Furthermore, as a special case of Meyers and Elcrat's paper [4] regarding the $2m$-th order variational equations subject to a mild elliptic structure, we can get the $W_{loc}^{m,2+\epsilon}(\Omega)$-estimate for some small $\epsilon > 0$ on