JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **37**, No. 3, pp. 251-262

doi: 10.4208/jpde.v37.n3.2
September 2024

Solution of Klein-Gordon Equation by a Method of Lines Using Reproducing Kernel Hilbert Space Method

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Received 17 September 2021; Accepted 11 May 2023

Abstract. This paper presents a method of lines solution based on the reproducing kernel Hilbert space method to the nonlinear one-dimensional Klein-Gordon equation that arises in many scientific fields areas. Our method uses discretization of the partial derivatives of the space variable to get a system of ODEs in the time variable and then solve the system of ODEs using reproducing kernel Hilbert space method. Consider two examples to validate the proposed method. Compare the results with the exact solution by calculating the error norms L_2 and L_∞ at various time levels. The results show that the presented scheme is a systematic, effective and powerful technique for the solution of Klein-Gordon equation.

AMS Subject Classifications: 65D15, 65Q05, 68Q25, 68U20

Chinese Library Classifications: O2

Key Words: Method of Lines; reproducing kernel Hilbert space method; Klein-Gordon equation.

1 Introduction

Nonlinear partial differential equations (PDEs) are extremely significant in engineering and research. Since most physical phenomena are nonlinear in nature and are represented by nonlinear PDEs. The one-dimensional hyperbolic nonlinear PDE proposed by Klein [1] and Gordon [2] is known as Klein-Gordon equation which is a relativistic wave equation, related to the Schrödinger equation.

Many authors have presented numerous approaches for solving the nonlinear Klein-Gordon equation in the recent few decades. For example, Morawetz [3] used time decay for the nonlinear Klein-Gordon equation. Strauss and Vazquez [4] evaluated numerical

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solution of the nonlinear Klein-Gordon equation. Ginibre and Giorgio [5] investigated the existence result of weak global solutions of the nonlinear Klein-Gordon equation in the energy space that can be obtained by the compactness method. Four explicit finite difference schemes were used to integrate the nonlinear Klein-Gordon equation by Jiménez and Vázquez [6] whereas Duncan [7] applied three finite difference approximations of the nonlinear Klein-Gordon equation and show that they are directly related to symplectic mappings. El-Sayed [8] used the decomposition method for studying the Klein-Gordon equation. Dehghan and Shokri [9] proposed a numerical scheme to solve the one-dimensional nonlinear Klein-Gordon equation with quadratic and cubic nonlinearity. Ikot et al. [10] presented an approximate solution of the Klein-Gordon equation for the Hulthén potential with equal scalar and vector potential.

This paper presents a semi-analytical method of lines (MOL) solution. The MOL was applied to solve the PDEs in (2001) by Schiesser et al. [11]. Schiesser's method is a technique based on a fully numerical scheme. In 2004, Koto [12] applied this technique to approximations of delay differential equations using Runge-Kutta method. Hamdi et al. (2009) [13] gave a basic idea of MOL. The semi-analytic MOL is used actively for solving linear PDE. For example see [14–16].

This work uses a different approach, employing the usual finite difference scheme for spatial discretization in the nonlinear initial boundary valued PDE to convert nonlinear initial valued system of Ordinary differential equations (ODEs) and then using reproducing kernel Hilbert space method (RKHSM) to get the solution. The theory of reproducing kernels dates to the first half of the 20th century, and its roots go back to the pioneering papers by S. Zeremba [17], Mercer [18], and Bergman [19–23]. In 1950, N. Aronszajn [24] outlined the past works and gave a systematic reproducing kernel theory and laid a good foundation for the research of each special case and greatly simplified the proof. This theory has been successfully implemented on linear and nonlinear applications with different type conditions by many authors [25–33]. The main idea is to construct the reproducing kernel space [34] absorb the conditions for determining solution of the nonlinear system of ODEs and represent the analytic solution is represented in the form of series. The RKHSM is easily implemented, grid-free and without time discretization. Also, we can evaluate the solution for finite number of points and use it often. In this article, we consider the following one-dimensional Klein-Gordon equation is given by

$$\frac{\partial^2 u}{\partial t^2}=\alpha\frac{\partial^2 u}{\partial x^2}+F(x,t,u), \qquad (x,t)\in\Omega\times\Gamma, \tag{1.1}$$

where $u=u(x,t)$ represents the wave displacement at position x and time t, $F(x,t,u)$ is the nonlinear force and α is a constant, with initial and boundary conditions

$$u(x,0)=\Theta_0(x), \quad u_t(x,0)=\Theta_1(x), \qquad x\in\overline{\Omega}, \tag{1.2}$$

$$u(a,t)=\Phi_1(t), \quad u(b,t)=\Phi_2(t), \qquad t\in\overline{\Gamma}, \tag{1.3}$$

where $\Omega=(a,b)$, $\Gamma=(0,T]$ with $0<T<\infty$.