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Blow-Up of Solution and Energy Decay for a Quasilinear Parabolic Problem

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Abstract. In this paper, we obtain the blow-up result of solutions and some general decay rates for a quasilinear parabolic equation with viscoelastic terms

$$A(t)|u_t|^{m-2}u_t-\Delta u+\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s=|u|^{p-2}u\log|u|.$$

Due to the presence of the log source term, it is not possible to use the source term to dominate the term $A(t)|u_t|^{m-2}u_t$. To bypass this difficulty, we build up inverse Hölder-like inequality and then apply differential inequality argument to prove the solution blows up in finite time. In addition, we can also give a decay rate under a general assumption on the relaxation functions satisfying $g'\leq -\xi(t)H(g(t))$, $H(t)=t^\nu$, $t\geq 0$, $\nu>1$. This improves the existing results.

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Key Words: Viscoelastic term; blow up; decay estimate.

1 Introduction

In this work, we consider the following problem

$$\begin{cases} A(t)|u_t|^{m-2}u_t-\Delta u+\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s=|u|^{p-2}u\log|u|, & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

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where $m\geq 2, T>0$, Ω is a bounded domain of $\mathbb{R}^n$, $n\in\mathbb{N}^*:=1,2,\cdots$, with a smooth boundary $\partial\Omega$, $g:\mathbb{R}^+\to\mathbb{R}^+$ is a positive nonincreasing function, and

$$A:\mathbb{R}^+\to M_n(\mathbb{R})$$

is a bounded positive definite matrix satisfying $A\in C(\mathbb{R}^+)$, that is, there exists a positive constant $c_0>0$ such that

$$(A(t)\vec{v},\vec{v})\geq c_0|\vec{v}|^2, \qquad \forall t\in\mathbb{R}^+,\ \forall\vec{v}\in\mathbb{R}^n, \tag{1.2}$$

where $(\cdot,\cdot)$ and $|\cdot|$ are the inner product and the norm $|\vec{v}|^2=\sum_{i=1}^n|v_i|^2$, $\vec{v}=(v_1,v_2,\cdot,v_n)$, respectively. In recent years, viscoelastic problems have been studied by many authors, particularly, the study of asymptotic behavior profile is more and more popular. For example, in 1996, Pucci P. and Serrin J. [1] considered systems of the following form

$$\begin{cases} A(t)|u_t|^{m-2}u_t=\triangle u-V(x)|u|^{p-2}u, & (x,t)\in\Omega\times(0,T),\\ u(t,x)=0, & (x,t)\in\partial\Omega\times(0,T), \end{cases}$$

where $1<p<\infty$, $1<m<\max\{p,\frac{2n}{n-2}\}$. They analyzed the dependence of the asymptotic stability of solutions on the exponents m,p and potential function $V(x)$. Later, in 2006, Messaoudi [2] studied an initial boundary value problem consisting of the equation

$$u_t-\Delta u+\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s=|u|^{p-2}u,$$

and proved if relaxation function g and the exponent p satisfied

$$2<p\leq\frac{2(n-1)}{n-2}, \qquad n>2,\ p>2,\ n=1,2,$$

then the phenomenon of blow-up occurred for positive initial energy. Motivated by Messaoudi' work, Liu and Chen [3] considered the equation

$$A(t)|u_t|^{m-2}u_t-\Delta u+\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s=|u|^{p-2}u,$$

for $m\geq 2$, $p\geq 2$, and the relaxation function g satisfying $g(0)>0$, $1-\int_0^\infty g(s)\mathrm{d}s=l>0$ and $g'\leq-\xi(t)g(t)$. They obtained a general decay of the energy function for the global solution and a blow-up result for the solution. Later, Youkana, Messaoudi and Guesmia [4] obtained some decay estimates for the following problem

$$A(t)|u_t|^{m-2}u_t-\Delta u+\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s=0,$$

under the assumption that g satisfying $g'\leq-\xi(t)g^p(t)$ with $p\in[1,\frac{3}{2})$. Recently, Youkana and Messaoudi [5] extended the related results to the general case when the relaxation function satisfies

$$g'(t)\leq-\xi(t)g(t)^p, \qquad \forall t\geq 0,\ p>1,$$