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Global and Nonglobal Solutions for Pseudo-Parabolic Equation with Inhomogeneous Terms

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Abstract. This paper considers the Cauchy problem of pseudo-parabolic equation with inhomogeneous terms $u_t=\Delta u+k\Delta u_t+w(x)u^p(x,t)$. In [1], Li et al. gave the critical Fujita exponent, second critical exponent and the life span for blow-up solutions under $w(x)=|x|^\sigma$ with $\sigma>0$. We further generalize the weight function $w(x)\sim|x|^\sigma$ for $-2<\sigma<0$, and discuss the global and non-global solutions to obtain the critical Fujita exponent.

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Key Words: Pseudo-parabolic equation; critical Fujita exponent; global solutions; blow-up.

1 Introduction

In this paper, we consider the following semilinear pseudo-parabolic equation

$$\begin{cases} u_t=\Delta u+k\Delta u_t+w(x)u^p(x,t), & (x,t)\in\mathbb{R}^n\times(0,T),\\ u(x,0)=u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $p>1$, $k>0$. The coefficient function $w(x)$ is continue and equals to $|x|^\sigma(-2<\sigma<0)$ for $|x|$ large enough, that is to say, there exist constants C_1, C_2, R_0, such that

$$C_1|x|^\sigma\le w(x)\le C_2|x|^\sigma,\qquad \text{for } |x|>R_0. \tag{1.2}$$

The initial data u_0 is a nontrivial nonnegative and appropriately smooth function.

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Pseudo-parabolic equation is a non-classical diffusion equation, possessing the third-order viscous term $k\Delta u_t$ [2,3]. These equations were studied in the existence, uniqueness and regularity for solutions, etc. We can refer to [4–16] and references therein.

To Cauchy problem of Pseudo-parabolic equation

$$u_t - k\Delta u_t = \Delta u + u^p(x,t), \qquad (x,t)\in\mathbb{R}^n\times(0,T).$$

Cao [17] studied the necessary existence, uniqueness, and comparison principle and proved the global existence and non-global existence results to obtain the critical Fujita exponent $p_c = 1+\frac{2}{n}$.

Recently, Li and Du [1] considered the pseudo-parabolic equation with inhomogeneous terms

$$\begin{cases} u_t - k\Delta u_t = \Delta u + |x|^\sigma u^p, & (x,t)\in\mathbb{R}^n\times(0,T), \\ u(x,0)=u_0(x), & x\in\mathbb{R}^n, \end{cases}$$

for $\sigma>0$. They researched the influence of the coefficient function $|x|^\sigma$ to the asymptotic behavior of solutions $u(x,t)$ and gave the critical Fujita exponent $p_c = 1+\frac{2+\sigma}{n}$.

The present paper generalizes the above problem with coefficient function $|x|^\sigma$ to $-2<\sigma<0$ for large $|x|$ by studying the problem (1.1). By dealing with the global and non-global existence of $u(x,t)$, we obtain the critical Fujita exponent $p_c = 1+\frac{2+\sigma}{n}$. This can be described by the following theorems.

Theorem 1.1. *For $1<p\le p_c$, any nonnegative solution $u(x,t)$ of* (1.1) *blow up in finite time.*

Theorem 1.2. *For $p>p_c$, there is a global solution of* (1.1) *with a small initial time.*

Remark 1.1. For $-2<\sigma<0$, the previously estimate $\||x|^\sigma \mathcal{G}_k(t)\varphi\|_{L^q(\mathbb{R}^n)}$ is invalid in Lemma 2.1 of [1]. In this paper, by dividing $\int_0^t \|w^{\frac{1}{p-1}}(x)\mathcal{G}_k(t-\tau)\mathcal{B}_k(w(x)u^p)\|_{L^p}\mathrm{d}\tau$ into $\int_0^{\frac{t}{2}} \|w^{\frac{1}{p-1}}(x)\mathcal{G}_k(t-\tau)\mathcal{B}_k(w(x)u^p)\|_{L^p}\mathrm{d}\tau$ and $\int_{\frac{t}{2}}^t \|w^{\frac{1}{p-1}}(x)\mathcal{G}_k(t-\tau)\mathcal{B}_k(w(x)u^p)\|_{L^p}\mathrm{d}\tau$, we deal with them by Lemmas 3.1 and 3.2 respectively.

This paper is organized as follows. Section 2 is dedicated to the non-global existence of the Cauchy problem (1.1) to prove Theorem 1.1. And, Section 3 deals with global solutions to prove Theorem 1.2.

2 Non-global solution

In this section, we consider the non-global solution of (1.1) to prove Theorem 1.1. First, we introduce the following argument.