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Regularity and Convergence for the Fourth-Order Helmholtz Equations and an Application

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Abstract. We study the regularity and convergence of solutions for the n-dimensional ($n=2,3$) fourth-order vector-valued Helmholtz equations

$$\mathbf{u}-\beta\Delta\mathbf{u}+\gamma(-\Delta)^2\mathbf{u}=\mathbf{v} \tag{VFHE}$$

for a given $\mathbf{v}$ in several Sobolev spaces, where $\beta>0$ and $\gamma>0$ are two given constants. By making use of the Fourier multiplier theorem, we establish the regularity and the L^p-L^q estimates of solutions for Eq. (VFHE) under the condition $\mathbf{v}\in L^p(\mathbb{R}^n)$. We then derive the convergence that a solution $\mathbf{u}$ of Eq. (VFHE) approaches $\mathbf{v}$ weakly in $L^p(\mathbb{R}^n)$ and strongly in $L^q(\mathbb{R}^n)$ as the parameter pair (β,γ) approaches $(0,0)$. In particular, as an application of the above results, for $(\mathbf{v},\mathbf{u})$ solving the following viscous incompressible fluid equations

$$\begin{cases}\mathbf{v}_t+\mathbf{u}\cdot\nabla\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{u}^T+\nabla p=\nu\Delta\mathbf{v},\\ \text{div }\mathbf{v}=\text{div }\mathbf{u}=0,\end{cases} \tag{INS}$$

we gain the strong convergence in $L^\infty([0,T],L^s(\mathbb{R}^n))$ from the Eqs. (VFHE)-(INS) to the Navier-Stokes equations as the parameter pair (β,γ) tending to $(0,0)$, where $s=\frac{2h}{h-2}$ with $h>n$.

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1 Introduction

For $v=\Gamma|u|^{p-2}u$ $(p>2)$ and $\gamma=1$, Bonheure, Casteras and Mandel in [1] previously adopted the dual method to establish the regularity of solutions for a fourth-order nonlinear Helmholtz equation:

$$\alpha u-\beta\Delta u+\gamma(-\Delta)^2u=v, \tag{FOHE}$$

under three cases: (a) $\alpha<0$, $\beta\in\mathbb{R}$ or (b) $\alpha>0$, $\beta<-2\sqrt{\alpha}$ or (c) $\alpha=0$, $\beta<0$. But for the case of $\alpha>0$ and $\beta>0$, there has no any result concerning the regularity of solutions for Eq. (FOHE). Without loss of generality, we study here the following fourth-order equations with vector-valued unknowns in $\mathbb{R}^n$ $(n=2,3)$:

$$\mathbf{u}-\beta\Delta\mathbf{u}+\gamma(-\Delta)^2\mathbf{u}=\mathbf{v}, \tag{1.1}$$

where $\alpha=1>0$, β and γ are two given positive parameters $(\beta>0,\ \gamma>0)$. In particular, for $(\mathbf{v},\mathbf{u})$ solving the viscous incompressible Navier-Stokes type equations

$$\begin{cases}\mathbf{v}_t+\mathbf{u}\cdot\nabla\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{u}^T+\nabla p=\nu\Delta\mathbf{v},\\ \text{div }\mathbf{v}=\text{div }\mathbf{u}=0,\end{cases} \tag{1.2}$$

(1.1)-(1.2) are generally called as Camassa-Holm equations with a fourth-order Helmholtz operator. This kind of equations with a second-order Helmholtz operator ((1.1)-(1.2) with $\gamma=0$) was first derived by Camassa and Holm [2] in 1993 in the study of shallow water equations. When $\gamma=0$, β is the parameter in the filter equations [2,3]. In order to regularize and stabilize the solutions of the second-order Helmholtz equations

$$\mathbf{u}-\beta\Delta\mathbf{u}=\mathbf{v},$$

(1.1) was introduced, where $\gamma>0$ can be viewed as a regularizing parameter in some sense (see also [1]).

There have been some results on the Helmholtz type equations [1, 3–7]. In particular, Bonheure, Casteras and Mandel in [1] as well as Bonheure and Nascimento [3] established the existence and the qualitative properties of solutions to a mixed dispersion fourth-order nonlinear Helmholtz type equation, where the dual method used by Ev*é*quoz and Weth in [6] was applied to establish the existence and regularity of solutions to these Helmholtz equations On the other hand, using Calder*o*n-Zygmund theorem and some estimates on the heat kernel, the existence and regularity for the second-order Helmholtz equations can be easily obtained which was mentioned in [8].

The aim in this paper is to investigate the regularity of the fourth-order Helmholtz equations (1.1) with the help of the Fourier multiplier theorem. Based on the fourth-order structure, we establish the regularity as well as some a priori estimates concerning the L^p-L^q estimates for the fourth-order Helmholtz equations (1.1). We then derive the convergence that a solution $\mathbf{u}$ of the equations (1.1) approaches $\mathbf{v}$ weakly in $L^p(\mathbb{R}^n)$ and