JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **37**, No. 3, pp. 326-354
doi: 10.4208/jpde.v37.n3.7
September 2024

Existence and Approximation of Statistical Solutions of the 3D MHD Equations

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Received 15 November 2022; Accepted 1 August 2023

Abstract. This paper focuses on the statistical characteristics of the 3D MHD equations. We firstly establish an existence theorem of a Vishik-Fursikov measure of the 3D MHD equations by taking advantage of the Krein-Milman theorem along with some functional and measure theories. Then by applying the Topsoe lemma on the constructed trajectory space possessing some special topological properties, we show that the Vishik-Fursikov measure and the stationary Vishik-Fursikov statistical solution of the 3D MHD system are approximated by the counterparts of the 3D MHD-α system, respectively, as the parameter α decreases to zero.

AMS Subject Classifications: 35Q35, 60B05, 76D06, 28A33

Chinese Library Classifications: O175.29

Key Words: 3D MHD equations; Vishik-Fursikov measures; stationary statistical solutions; convergence of measures.

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1 Introduction

The magnetohydrodynamic (MHD) equations describe the motion of electrically conducting fluids arising from electrolytes, plasmas or liquid metals. The essential physical principle of the MHD equations is that magnetic fields induce currents in a moving conductive fluid, which in turn polarize the fluid and reciprocally change the magnetic field itself [1–4].

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This paper investigates the 3D MHD equations

$$\begin{cases} \partial_t u+(u\cdot\nabla)u-\nu\Delta u+\nabla\pi+\dfrac{1}{2}\nabla|B|^2=(B\cdot\nabla)B+g(x), & x\in\Omega,\ t\in\mathbb{R}_+, \\ \partial_t B+(u\cdot\nabla)B-(B\cdot\nabla)u-\eta\Delta B=0, & x\in\Omega,\ t\in\mathbb{R}_+, \\ \nabla\cdot u=0, \quad \nabla\cdot B=0, & x\in\Omega,\ t\in\mathbb{R}_+, \\ \displaystyle\int_\Omega u(x,t)\mathrm{d}x=0, \quad \int_\Omega B(x,t)\mathrm{d}x=0, & t\in\mathbb{R}_+, \end{cases} \tag{1.1}$$

with the periodic boundary conditions

$$\begin{aligned} u(x_1,x_2,x_3)=u(x_1+L,x_2,x_3)=u(x_1,x_2+L,x_3)=u(x_1,x_2,x_3+L), \\ B(x_1,x_2,x_3)=B(x_1+L,x_2,x_3)=B(x_1,x_2+L,x_3)=B(x_1,x_2,x_3+L), \end{aligned}$$

and the initial condition

$$(u(x,0),B(x,0))=(u_0(x),B_0(x)), \quad x\in\Omega.$$

Here the variable $x:=(x_1,x_2,x_3)$ is in $\Omega:=[0,L]^3$ with $L>0$, and $\mathbb{R}_+:=[0,+\infty)$. The functions $u(x,t)=(u^1(x,t),u^2(x,t),u^3(x,t))$, $B(x,t)=(B^1(x,t),B^2(x,t),B^3(x,t))$ and π represent the fluid velocity field, the magnetic field and the scalar pressure at each point of the fluid, respectively. The parameter ν denotes the kinematic viscosity of the fluid , η means the magnetic diffusivity and g is a given periodic field of external forces.

Although many scholars made efforts to study the solution of the 3D MHD system [5–12], the uniqueness of the global solution of (1.1) still remains open [13]. Deugoue [14] studied the trajectory attractor of (1.1) which describes the long-time behavior of solutions as time approaches infinite. Being similar to to the trajectory attractor, the statistical solution is an effective tool for the study of evolution equations displaying the property of global existence of weak solutions without a known result of global uniqueness.

The first goal of this paper is to verify the existence of a Vishik-Fursikov measure of (1.1). The statistical solution was initially pioneered to study the Navier-Stokes equations which displays quite wild behavior in varying space and time. The statistical solution established by Foias and Prodi [15, 16] is a family of measures on phase space, parameterized by the time variable, that conforms to the Liouville equation and satisfies several regularity properties. Another statistical solution defined by Vishik and Fursikov [17–19] is one single probability measure on trajectory space. More recently, Foias, Rosa and Temam [20] elaborated a statistical solution (called Vishik-Fursikov measure), established an existence theorem making use of the Krein-Milman theorem [21], and proved that projecting a Vishik-Fursikov measure to the phase space, at each time, yields a statistical solution in the sense of Foias and Prodi. Some researchers developed abstract framework for the theory of statistical solutions of general evolution equations which possess properties similar to that of the 3D Navier-Stokes equations [22–27].