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$\mathcal{L}^{2,\mu}(Q)$-ESTIMATES FOR PARABOLIC EQUATIONS AND APPLICATIONS[1]

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Abstract In this paper we derive *a priori* estimates in the Campanato space $\mathcal{L}^{2,\mu}(Q_T)$ for solutions of the following parabolic equation

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j} + a_i u) + b_i u_{x_i} + cu = \frac{\partial}{\partial x_i}f_i + f_0$$

where $\{a_{ij}(x,t)\}$ are assumed to be measurable and satisfy the ellipticity condition. The proof is based on accurate DeGiorgi-Nash-Moser's estimate and a modified Poincare's inequality. These estimates are very useful in the study of the regularity of solutions for some nonlinear problems. As a concrete example, we obtain the classical solvability for a strongly coupled parabolic system arising from the thermistor problem.

Key Words Parabolic equation; *a priori* estimates in Campanato space; DeGiorgi-Nash-Moser's estimate; a modified Poincare's inequality.

Classification 35K20, 35K55.

1. Introduction

Let Ω be a bounded domain in R^n with boundary $S = \partial\Omega$ in C^1 and $Q_T = \Omega \times (0, T]$ with $T > 0$. Consider the following parabolic equation:

$$u_t - \frac{\partial}{\partial x_i}(a_{ij}(x,t)u_{x_j} + a_i u) + b_i u_{x_i} + cu = \frac{\partial}{\partial x_i}f_i + f_0 \tag{1.1}$$

where a_{ij} satisfies the ellipticity condition:

$$a_0|\xi|^2 \le a_{ij}\xi_i\xi_j \le A_0|\xi|^2 \quad \text{for } \xi \in R^n, \quad 0 < a_0 \le A_0$$

It is well known that the DeGiorgi-Nash-Moser estimate plays an essential role in the study of solvability for nonlinear parabolic equations. However, this estimate is often not enough in dealing with regularity of solutions. On the other hand, the theory of the

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Campanato space $\mathcal{L}^{2,\mu}$ is powerful for investigating regularity of solutions for elliptic equations and systems (cf. [1], [2], etc.). In the present work we would like to derive the $\mathcal{L}^{2,\mu}(Q)$-estimates for weak solutions of the equation (1.1). It will be seen the results are also very useful in applications. The core of the proof is based on accurate DeGiorgi-Nash-Moser's estimates. For elliptic equations, the theory can be found in [2]. The fundamental difference from the elliptic theory is that Poincare's inequality

$$\int_{Q_r}(u-u_{z_0,r})^2dxdt \le Cr^2\int_{Q_{2r}}|\nabla u|^2dxdt$$

does not hold for a general function $u(x,t)\in L^2(0,T;H^1(\Omega))$ (see the notation below). However, by using the equation and combining (elliptic version) Poincare's inequality, we are able to resolve the difficulty. The proof is based on various modifications of elliptic situation.

For convenience we introduce some standard notations: a point (x,t) in Q_T will be denoted by z. The distance between two points $z_1=(x_1,t_1)$ and $z_2=(x_2,t_2)$ is equal to

$$\max\{|x_1-x_2|,|t_1-t_2|^{\frac{1}{2}}\}$$

For $r>0$,

$$B_r(x_0)=\{x\in R^n : |x-x_0|<r\} \text{ and } Q_r(z_0)=B_r(x_0)\times(t_0-r^2,t_0]$$

For a measurable set $A\subset R^n\times[0,T]$ with a finite measure $|A|<\infty$,

$$\fint_A udz=\frac{1}{|A|}\int_A udz$$

In particular, when $A=Q_r(z_0)$,

$$u_{z_0,r}=\fint_{Q_r(z_0)}udz$$

For $\mu>0$, let

$$[u]_{2,\mu,Q_r}=\left(\sup_{z_0\in Q,0<\rho<r}\rho^{-\mu}\int_{Q_\rho(z_0)}|u-u_{z_0,\rho}|^2dz\right)^{\frac{1}{2}}$$

The space $\mathcal{L}^{2,\mu}(Q)$ consists of all functions in $L^2(Q)$ such that

$$[u]_{2,\mu,Q_r}<\infty$$

We understand that $Q\cap Q_r$ should be used in the integration whenever Q_r is not a subset of Q. $\mathcal{L}^{2,\mu}(Q)$ is a Banach space with the norm

$$\|u\|_{2,\mu,Q_r}=\{\|u\|^2_{L^2(Q_r)}+[u]^2_{2,\mu,Q_r}\}^{\frac{1}{2}}$$