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ON THE ZEROS AND ASYMPTOTIC BEHAVIOR OF MINIMIZERS TO THE GINZBURG-LANDAU FUNCTIONAL WITH VARIABLE COEFFICIENT

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Abstract In this paper a partial answer to the fourth open problem of Bethuel-Brezis-Hélein [1] is given. When the boundary datum has topological degree ±1, the asymptotic behavior of minimizers of the Ginzburg-Landau functional with variable coefficient $\frac{1}{x_1}$ is given. The singular point is located.

Key Words Ginzburg-Landau functional; asymptotics; vortices.
Classification 35J55, 35Q40.

1. Introduction

Recently, Bethuel-Brezis-Hélein [1-3] have studied the asymptotic behavior for the minimizers u_ε of the following Ginzburg-Landau functional in $H_g^1(\Omega; R^2) \equiv \{v \in H^1(\Omega, R^2), u\mid_{\partial\Omega} = g\}$,

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right] \tag{1.1}$$

where Ω is a simply connected, star-shaped and bounded smooth domain in R^2, $g : \partial\Omega \to S^1$ is a smooth map, ε is a small parameter. They proved that there is a subsequence $\varepsilon_n \downarrow 0$ such that

$$u_{\varepsilon_n} \to u_* \text{ in } C^{1+\alpha}_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \cdots, a_{|d|}\}) \text{ and in } C^k_{\text{loc}}(\Omega), \quad \forall k \in \mathbb{N}$$

where $d = \deg(g, \partial\Omega)$ denotes the winding number, $u_* : \Omega\backslash\{a_1, \cdots, a_{|d|}\} \to S^1$ is a smooth harmonic map, $a_1, \cdots, a_{|d|}$ are the limit positions of the zeros of u_{ε_n} (zeros of u_{ε_n} are called vortices which correspond to the normal points in superconductor) which minimize the so-called renormalized energy $W(b)$ (see [1]). This problem is related to the phase transition in superconductivity (see [4]).

In their proofs, a key estimate

$$\frac{1}{\varepsilon^2}\int_\Omega (1-|u_\varepsilon|^2)^2 \le C \tag{1.2}$$

is derived from the global Pohozaev identity. From (1.2), for ε small enough, one can obtain the uniform upper bound on the number of zeros of u_ε. Then the precise lower and upper bounds on the energy $E_\varepsilon(u_\varepsilon)$ lead to *a priori* estimate for u_ε in $H^1_{\rm loc}(\Omega\backslash\{a_1,\cdots a_{|d|}\})$. Finally, they obtained the convergence of u_{ε_n}, subsequence of minimizers, in various norms.

In [5], based on a local version of (1.2), M. Struwe got a similar result to [1] without the restriction of star-shapedness on Ω. There are also some other generations (see [6–10]).

In this paper, we discuss open Problem 4 in [1]. That is,

$$E_\varepsilon(u_\varepsilon)=\frac{1}{2}\int_\Omega \frac{1}{x_1}\left[|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)^2\right] \tag{1.3}$$

where $\Omega=\{(x_1,x_2)\in R^2|(x_1-1)^2+x_2^2<R^2, 0<R<1\}$, $u_\varepsilon\in H^1_g(\Omega,R^2)$, g is as above. We intend to study the behaviour of minimizers u_{ε_n} as $\varepsilon_n\downarrow 0$. This problem is related to the model of superconducting thin films having variable thickness (see [11]). In contrast with [1], we call our problem Ginzburg-Landau model with variable coefficient.

In our case, some arguments in [1] or [5] do not work. As a try, we only consider a special situation, i.e., $\deg(g,\partial\Omega)=\pm1$. By a different way, we prove that u_ε has unique zero (in Section 3). To get a uniform estimate, we use Lemma 4.4 to prove that $|u_\varepsilon|\ge\frac{1}{2}$ in $\overline{\Omega}\backslash B(x_\varepsilon,2\varepsilon^{\beta_1})$, $0<\beta_1<1/2$, x_ε is the unique zero of u_ε. This is much different from [1] in which they prove $|u_\varepsilon|\ge\frac{1}{2}$ in $\overline{\Omega}\backslash B(x_\varepsilon,\lambda_0\varepsilon)$. Next, we prove that $x_\varepsilon\to a=(1+R,0)$ and for any sequence u_{ε_n}, there is a subsequence, still denoted by u_{ε_n}, such that $u_{\varepsilon_n}\to u_*$ in $C^k(K)$, $\forall k\in\mathbb{N}$, $\forall K\subset\subset\Omega$, where u_* is a harmonic map from $\Omega\to S^1$. The Euler equation of (1.3) is

$$\begin{cases}-\Delta u_\varepsilon+\dfrac{1}{x_1}u_{\varepsilon x_1}=\dfrac{1}{\varepsilon^2}u_\varepsilon(1-|u_\varepsilon|^2) & \text{in } \Omega\\ u_\varepsilon|_{\partial\Omega}=g\end{cases} \tag{1.4}$$

This paper is organized as follows. In Section 2, we shall discuss the case $\deg(g,\partial\Omega)=0$ which is the base of the case $|\deg(g,\partial\Omega)|=1$; In Section 3, we prove the existence and uniqueness of the zero of u_ε; In Section 4, through a series of *a priori* estimates, we establish the asymptotic behavior of u_ε, i.e., Theorem 4.1, our main result.