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HYPERBOLIC PHENOMENA IN A DEGENERATE PARABOLIC EQUATION

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Abstract M. Bertsch and R. Dal Passo [1] considered the equation $u_t = (\varphi(u)\psi(u_x))_x$, where $\varphi > 0$ and ψ is a strictly increasing function with $\lim_{s\to\infty}\psi(s) = \psi_\infty < \infty$. They have solved the associated Cauchy problem for an increasing initial function. Furthermore, they discussed to what extent the solution behaves like the solution of the first order conservation law $u_t = \psi_\infty(\varphi(u))_x$. The condition $\varphi > 0$ is essential in their paper. In the present paper, we study the above equation under the degenerate condition $\varphi(0) = 0$. The solution also possesses some hyperbolic phenomena like those pointed out in [1].

Key Words Degenerate parabolic equation; entropy condition.
Classification 35K65.

1. Introduction

We consider the problem

$$\text{(I)}\qquad \begin{cases} u_t = (\varphi(u)\psi(u_x))_x, & x \in R, t \in (0,\infty) \\ u(x,0) = u_0(x), & x \in R \end{cases}$$

where $\varphi : R^+ \to R^+$ is smooth, $\varphi \in C[0,+\infty)$, $\varphi(0) = 0$, $\varphi'(s) > 0$ $(s > 0)$ and $\lim_{s\to 0} +\frac{s}{\varphi(s)} = 0$. $\psi : R \to R$ is a smooth, odd function such that $\psi' > 0$ in R and $\lim_{s\to+\infty}\psi(s) = \psi_\infty$.

The initial function $u_0 : R \to R$ is bounded, strictly increasing and

$$\lim_{x\to-\infty} u_0(x) = 0, \quad \lim_{x\to+\infty} u_0(x) = A \tag{1}$$

$$u_0'(x) = O(u_0(x)) \quad \text{as } x \to -\infty \tag{2}$$

$$u_0'(x) = O(A - u_0(x)) \quad \text{as } x \to +\infty \tag{3}$$

For the construction of a solution we use a standard parabolic regularization: Let $\varepsilon > 0$ and u_ε be the unique smooth solution of the problem

$$(\mathrm{I}_\varepsilon)\qquad \begin{cases} u_t = (\varphi_\varepsilon(u)\psi_\varepsilon(u_x))_x, & x \in R, t \in (0,\infty) \\ u(x,0) = u_{0\varepsilon}(x), & x \in R \end{cases}$$

where u_ε is a smooth approximation of u_0 and

$$\varphi_\varepsilon(s) = \varphi(s+\varepsilon), \quad s \in [0,\infty) \tag{4}$$

$$\psi_\varepsilon(s) = \psi(s) + \varepsilon s, \quad s \in R \tag{5}$$

We shall show that

$$u_{\varepsilon_i} \to u \quad \text{in } L^1_{\text{loc}}(R \times [0,\infty)) \quad as\ i \to \infty \tag{6}$$

for some sequence $\varepsilon_i \to 0$ as $i \to \infty$, and

$$u \in L^\infty(R \times R^+) \cap BV_{\text{loc}}(R \times [0,\infty))$$

The main results of this paper can be stated as:

If u_0 is strictly increasing and satisfies (1)–(3), then

(i) u (defined by (6)) is a solution of Problem I.

(ii) u is not necessary to be continuous, even if u_0 is continuous.

(iii) $\psi(u_x)$ is a continuous function (under the convention that $\psi(\infty) = \lim_{s\to\infty} \psi(s) = \psi_\infty$).

(iv) u satisfies an entropy-type condition: if at some point $(x,t) \in R \times R^+$

$$u^+ = u(x^+,t) > u^- = u(x^-;t)$$

then

$$\varphi(s) \le \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-}(s - u^-) + \varphi(u^-) \quad \text{for } s \in [u^-, u^+]$$

(v) The entropy condition is necessary for uniqueness of solutions, i.e., there may exist solutions which do not satisfy the entropy condition.

(vi) Let $C_1 \le u(x,t) \le C_2$ for $(x,t) \in D = (x_1,x_2) \times (t_1,t_2)$ for some $C_1 \le C_2$, $x_1 < x_2$, $0 < t_1 < t_2$. Moreover, if φ is strictly concave in $[C_1, C_2]$, then $u_x \in L^\infty_{\text{loc}}(D)$.

The most striking results are the points (ii), (iv) and (v) which show the hyperbolic character of Problem I. These results are, however, expectable. Because the parabolicity of equation in Problem I is so weak for $u_x \to \infty$ that the solution may become discontinuous and behave like the solution of first-order equation $u_t = \psi_\infty(\varphi(u))_x$ (i.e., the discontinuity satisfies Rankine-Hugoniot condition and the entropy condition).

In order to prove the result which concerns the behaviour of the level curves ("characteristic") of u near a shock front, we need a further assumption

$$\psi'(s) \ge cs^{-2} \quad \text{for } s \le s_0 > 0 \tag{7}$$