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A NEW PROOF OF HAMILTON'S THEOREM ON HARMONIC MAPS FROM MANIFOLDS WITH BOUNDARY

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Abstract We give a simple proof of the well-known Hamilton's result [1] on the heat flows and harmonic maps from manifolds with boundary using the approach of Ding-Lin [2].
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Let M be a compact Riemannian manifold with boundary ∂M, let N be a compact Riemannian manifold. We denote $M \cup \partial M$ by $\overline{M}$. Since N can be isometrically embedded into an Euclidean space $\mathbf{R}^k$ for some $k > n$, we may view N as a submanifold of $\mathbf{R}^k$.

For any $u \in C^1(M, N)$, the energy density of u is defined by $e(u) = \frac{1}{2}|\nabla u|^2$, the energy is given by $E(u) = \int_M e(u)dV$. The Euler-Lagrange equation associated with the energy functional is

$$\tau(u) \equiv \Delta u - A(u)(du, du) = 0$$

where Δ is the Laplace operator on M and $A(u)$ is the second fundamental form of N in $\mathbf{R}^k$ at u. By the definition, the solution of this equation is called harmonic.

Hamilton [1] proves the following theorem.

Theorem 1 *Let M be a compact Riemannian manifold with boundary, and let N be a compact Riemannian manifold with nonpositively sectional curvature. Let $h : \partial M \to N$ be any smooth map and $u_0 : \overline{M} \to N$ a smooth map with $u_0 |_{\partial M} = h$ in any given relative homotopy class. There exists a continuous map $u : \overline{M} \times [0, \infty) \to N$ smooth except at the corner $\partial M \times \{0\}$ satisfying the following heat equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tau(u) \\ u(\cdot, t) |_{\partial M} = h \\ u(x, 0) = u_0(x) \end{cases} \tag{1}$$

on $\overline{M} \times [0, \infty)$. For a suitable choice of a sequence $t_n \to \infty$ the maps $u(\cdot, t_n) : \overline{M} \to N$ converge in $C^\infty(\overline{M}, N)$ to a harmonic map $u_\infty : \overline{M} \to N$ in the same relative homotopy class as f_0.

It is proved by Hamilton that (1) has a unique solution $u(x,t)$ for $0 \le t < T \in (0, \infty]$ which is smooth except on the corner $\partial M \times \{0\}$. Here $T = T(u_0)$ is the maximal existence time for the solution u. To prove Theorem 1 it suffices to show that the solution of (1) satisfies the following *a priori* estimate

$$\| \nabla u(t) \|_{C^0(\overline{M})} \le C(E(u_0), \|u_0\|_{C^{2,\alpha}(\overline{M})}) \tag{2}$$

for $T > t \ge$ some $t_0 > 0$. When M has no boundary, Ding-Lin derive the existence of an m-obstruction under the assumption that (2) does not hold. They use the rescaling technique and the monotonicity inequality [3] for the heat equation. The main purpose of this note is to show that their ideas also work for the first initial-boundary value problem (1). However, in our case we have to deal with the solution near the boundary. Consequently we shall derive the existence of a more general m-obstruction (see the definition below) if (2) does not hold.

Definition 2 *Let $\mathbf{R}^m_+(-\delta) = \{(x', x^m) \in \mathbf{R}^m \mid x^m > -\delta\}$ where $0 \le \delta \le \infty$. We say that $v \in C^2(\overline{\mathbf{R}^m_+(-\delta)} \times (-\infty, 0], N)$ is an m-obstruction if*

(i) *it satisfies the heat equation*

$$\frac{\partial v}{\partial t} = \tau_0(v) \quad \text{on } \mathbf{R}^m_+(-\delta) \times (-\infty, 0] \tag{3}$$

(ii)

$$|\nabla v|(x,t) \le |\nabla v|(0,0) \ne 0, \quad \forall (x,t) \in \mathbf{R}^m_+(-\delta) \times (-\infty, 0] \tag{4}$$

(iii) *if $\delta < \infty$,*

$$v\,|_{\partial \mathbf{R}^m_+(-\delta)} \equiv Const \tag{5}$$

(iv) *there exists $E_0 > 0$, such that*

$$R^{2-m} \int_{B^+_R(-\delta)} |\nabla v(t)|^2 dV \le E_0 \tag{6}$$

where $B^+_R(-\delta) = \{x \in \mathbf{R}^m \,||x| < R, x^m > -\delta\}$.

In the proof of the existence of an m-obstruction, we also use the rescaling technique. It is natural that we will use the monotonicity inequality for the first initial-boundary value problem (1) derived by Chen [4] and Chen-Lin [5] instead of the one obtained by Struwe [3].