J.Partial Diff. Eqs. 10(1997), 213–231
©International Academic Publishers Vol.10, No.3

MUSKAT PROBLEM WITH SURFACE TENSION

Hong Jiaxing
(Institute of Mathematics, Fudan University, Shanghai 200433, China)
Tao Youshan
(Department of Basic Sciences, China Textile University, Shanghai 200051, China)
Yi Fahuai
(Department of Mathematics, Suzhou University, Suzhou 215006, China)
(Received July 8, 1995; revised Feb. 5, 1996)

Abstract In this paper, we consider Muskat problem with surface tension at the free boundary. Under the natural conditions, we prove the existence of classical solution locally in time by Schauder fixed point theorem.
Key Words Lower order terms; model problem; Fréchet derivative; pseudo-differential operator; J. Simon's compact theorem.
Classification 35R35.

1. Introduction

When two fluids in motion occupy a porous medium domain, we consider a simultaneous flow of two immiscible fluids or phases in the pore space, for example, consider a problem which models the extraction of oil from the ground by water. We assume that an abrupt interface separates the two fluids and that on the each side of the interface there only exists a single phase (fluid). In oil literature, the displacement of the two fluids is usually called piston-like. It is free boundary problem.

While two immiscible fluids are in contact in the interstices of a porous medium, a discontinuity in pressure exists across the interface separating two fluids. Its magnitude depends on the interface mean curvature at the point. The difference in pressure is called Capillary pressure p_c:

$$p_{nw} - p_w = p_c$$

where p_{nw} and p_w are the pressure in nonwetting and wetting phase respectively. And from Laplace equation for capillary pressure $p_c = \sigma k$, where σ is the surface tension and k is the mean curvature.

From the Law of conservation of mass and Darcy's Law, the problem (for incompressible fluids) is formulated as follows

$$(\mathrm{P})\begin{cases} -\nabla\cdot\left(\dfrac{k}{\mu_w}\nabla p_w\right)=0 & \text{in } Q_w(\text{water region})\\ -\nabla\cdot\left(\dfrac{k}{\mu_0}\nabla p_0\right)=0 & \text{in } Q_0(\text{oil region})\\ p_w-p_0=\sigma k=\sigma\nabla\cdot n & \text{on } \Gamma(\text{interface})\\ -\dfrac{k}{\mu_w}\nabla p_w\cdot n=-\dfrac{k}{\mu_0}\nabla p_0\cdot n=\phi v_n & \text{on } \Gamma \end{cases}$$

where k is the permeability, μ_0 and μ_w are viscosities of oil and water respectively. n is the normal of $\Gamma_t=\Gamma\cap\{t\}$, v_n is the velocity of advance in the normal direction of Γ_t, ϕ is the porosity.

If we neglect the capillary pressure p_c (i.e., $\sigma=0$), the problem (P) is called Muskat problem, which was proposed by Muskat in 1934 ([12]).

If $\sigma>0$, the problem (P) is called the Muskat problem with surface tension.

To our knowledge, there were no essential advances yet about Muskat problem and Muskat problem with surface tension.

The capillary force may affect the stability of the front in the physical fact. For the stability of the interface it is a good term, it always tends to stabilize the displacement front.

So we expect that the Muskat problem with surface tension is well-posed under the natural conditions. We shall see this from the following proof of the existence of classical solution for the problem.

The existence of classical solution local in time about problem (P) is equivalent to the existence of a zero point of nonlinear operator $\mathcal{F}$. The crucial step of our proof is to prove the invertibility of $\mathcal{F}'$ (—— Fréchet derivative of $\mathcal{F}$).

2. Formulation of the Problem

Let $G\subset\mathbf{R}^n$ $(n=1,2)$ be an open domain, define function spaces:

$$C_T^{k+\alpha}(\overline{G})=C([0,T],C^{k+\alpha}(\overline{G}))\quad(0<\alpha<1,\ k=1,2,\cdots)$$

$$\widehat{C}_T^{k+\alpha}(\overline{G})=\{v|v\in C_T^{k+\alpha}(\overline{G}),\partial_t v\in C_T^{k-3+\alpha}(\overline{G})\}\quad(k=4,5,\cdots)$$

$$\underset{\circ}{C}{}_T^{k+\alpha}(\overline{G})=\{v|v\in C_T^{k+\alpha}(\overline{G}),v\,|_{t=0}=0\}$$

$$\underset{\circ}{\widehat{C}}{}_T^{k+\alpha}(\overline{G})=\{v|v\in \widehat{C}_T^{k+\alpha}(\overline{G}),v\,|_{t=0}=\partial_t v\,|_{t=0}=0\}$$

$$\|v\|_{\widehat{C}_T^{k+\alpha}(\overline{G})}=\|v\|_{C_T^{k+\alpha}(\overline{G})}+\|\partial_t v\|_{C_T^{k-3+\alpha}(\overline{G})}$$