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THE SOLVABILITY OF THE BOUNDARY VALUE PROBLEM OF THE FOUR ELEMENTS WITH THE CARLEMAN SHIFT AND COMPLEX-CONJUGATE VALUES

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Abstract To following boundary value problem

$$a(t)\overline{\Phi^+(t)} + b(t)\Phi^+[\alpha(t)] + c(t)\overline{\Phi^-(t)} + d(t)\Phi^-[\alpha(t)] = f(t)$$

this paper has given out the number of linearly independent solutions and the conditions of solvability, so as to deepen the study of the problem to the same level as that of following problem

$$a(t)\overline{\Phi^+(t)} + b(t)\Phi^+(t) + c(t)\overline{\Phi^-(t)} + d(t)\Phi^-(t) = f(t)$$

Key Words Boundary value problem; Noether problem; number of linearly independent solutions; number of the conditions of solvability.

Classification 35Q15.

This paper discusses a boundary value problem on the unite circle $L : |z| = 1$ for the following equation:

$$a(t)\overline{\Phi^+(t)} + b(t)\Phi^+[\alpha(t)] + c(t)\overline{\Phi^-(t)} + d(t)\Phi^-[\alpha(t)] = f(t) \tag{1}$$

in which the unknown functions, $\Phi^+(z), \Phi^-(z)$ are analytical within D^+ and D^- respectively; and functions $a(t), b(t), c(t), d(t), f(t), \alpha'(t) \in H_\mu(L)$: where $\alpha(t)$ is the positive and negative shifts that satisfy conditions (K2) of Carleman[1].

A special equality can be expressed as follows:

$$a(t)\Phi^+(t) + b(t)\overline{\Phi^+(t)} + c(t)\Phi^-(t) + d(t)\overline{\Phi^-(t)} = f(t) \tag{2}$$

To the problem. Ref.[1] has given out a theorem as below.

Theorem 1 *To classify the problem (1) as the Noether problem needs the necessary and sufficient conditions as follows: when $\alpha(t)$ is a positive shift, $\theta(t) \neq 0$; when $\alpha(t)$ is a negative shift, $\theta_1(t) \neq 0$, $\theta_2(\theta) \neq 0$, where*

$$\theta(t) = a(t)\overline{c[\alpha(t)]} - \overline{b[\alpha(t)]}d(t)$$
$$\theta_1(t) = a(t)\overline{a[\alpha(t)]} - b(t)\overline{b[\alpha(t)]}$$
$$\theta_2(t) = c(t)\overline{c[\alpha(t)]} - d(t)\overline{d[\alpha(t)]}$$

If there exists $V(t) = b(t)\overline{c[\alpha(t)]} - \overline{a[\alpha(t)]}d(t)$, it is easy to testify the following identities

$$\theta(t)\overline{\theta[\alpha(t)]} - V(t)\overline{V[\alpha(t)]} = \theta_1(t)\theta_2(t) \tag{3}$$
$$\theta_1(t) = \overline{\theta_1[\alpha(t)]}, \quad \theta_2(t) = \overline{\theta_2[\alpha(t)]} \tag{4}$$

It is expected by the author of [1] that the study in (1) can reach the same depth as that of the problem (2). The reference has discussed such cases as $\alpha(t)$ is a positive shift and $\theta_1(t) = \theta_2(t) = 0$, and $\alpha(t)$ is a negative shift and $\theta(t) = 0$ and none of $\theta(t), \theta_1(t), \theta_2(t)$ is zero. To the former the reference has given out the number of linearly independent solutions and the conditions of solvability. This paper will study the problem (1) under the conditions $\theta_1 \neq 0$, $\theta_2 \equiv 0$ and $\theta_2 \neq 0$, $\theta_1 \equiv 0$ so as to deepen the study of problem (1) to the same level as that of the problem (2).

1. A Case that $\alpha(t)$ is a Positive Shift and $\theta_2(t) \equiv 0$

By making complex-conjugate and transferring of t into $\alpha(t)$, two equalities will be derived from (1). Then by eliminating $\overline{\Phi^+[\alpha(t)]}$ from the two equalities, a function which is equivalent to (1) will be derived as follows:

$$\overline{\theta_1(t)}\Phi^+(t) = -\theta[\alpha(t)]\Phi^-(t) + \overline{V(t)}\,\overline{\Phi^-[\alpha(t)]} + f_1(t) \tag{5}$$

where $f_1(t) = \overline{f(t)}a[\alpha(t)] - f[\alpha(t)]\overline{b(t)}$. In order to ensure that (1) belongs to the Noether problem, it is supposed $\theta(t) \neq 0$ and the following boundary value problem is to be considered

$$\Phi^+[\alpha(t)] = -\frac{\theta(t)}{V(t)}\overline{\Phi^+(t)} + \frac{\theta(t)\overline{f_1(t)} + V(t)f_1[\alpha(t)]}{\theta_1(t)V(t)} \tag{6}$$

and

$$\Phi^-[\alpha(t)] = \frac{\overline{\theta[\alpha(t)]}}{V(t)}\overline{\Phi^-(t)} + \frac{\overline{F(t)}}{V(t)} \tag{7}$$

where $F(t) = \overline{\theta_1(t)}\Phi^+(t) - f_1(t)$, $\Phi^+(z)$ is the solutions of the problem (6). According to (3), $V(t) \neq 0$.