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A REMARK ABOUT HYPERBOLIC EQUATIONS WITH LOG-LIPSCHITZ COEFFICIENTS

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Abstract We prove the well-posedness of the Cauchy problem for strictly hyperbolic equations and systems with Log-Lipschitz coefficients in the time variable.
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1. Introduction

In [1] Colombini and Lerner proved the well-posedness in C^∞ of the Cauchy problem for strictly hyperbolic second order operators whose coefficients are "Log-Lipschitz" in the time variable t, C^∞ in the space variables x. The purpose of this short paper is to extend this result to operators of higher order and to systems.

A function α of the variable t is said to be Log-Lipschitz if it satisfies

$$|\alpha(t_1) - \alpha(t_2)| \leq C|t_1 - t_2|\,|\log|t_1 - t_2||$$

for small $|t_1 - t_2|$.

For P an operator of order $d \geq 2$, Theorem 3 below gives the energy estimate

$$\sum_{j<d} \|\partial_t^j u(t)\|_{H^{m-j+d-1-\beta t}} \leq C_m\Big(\sum_{j<d} \|\partial_t^j u(0)\|_{H^{m-j+d-1}} + \int_0^t \|Pu(s)\|_{H^{m-\beta s}} ds\Big)$$

with $\beta > 0$, $0 \leq t \leq 1/\beta$. When $d = 2$, this result is the same as that of Theorem 2.2 and Theorem 2.3 in [1].

Since we use elementary arguments our proof is very easy but it does not work for isotropically Log-Lipschitz coefficients (i.e. with respect to all variables). So we are not able to extend either Theorem 2.1 in [1] by means of the technique of this paper.

When the coefficients satisfy an inequality of the type

$$|\alpha(t_1,x)-\alpha(t_2,x)|\le C|t_1-t_2||\phi(|t_1-t_2|)|,\quad \text{with } |\phi(r)|\to+\infty \text{ as } r\to 0^+$$

a formal use of our method suggests the well-posedness of the Cauchy problem in the spaces

$$H^m_\phi=\{u;(\exp|\phi(1/D_x)|)u\in H^m\}$$

giving a heuristic explanation of the following results:

i) The Log-Lipschitz regularity is a natural threshold beyond which the well-posedness in C^∞ could not be expected. An effective counterexample is given in [1];

ii) For $C^{0,k}$ coefficients, $0<k<1$, (i.e. for $\phi(r)=r^{k-1}$) the Cauchy problem is well-posed in the Gevrey classes $G^{(\sigma)}$, $\sigma<1/(1-k)$. We refer to [2] for a precise use of our method in this situation. There we re-obtain and extend some results of $G^{(\sigma)}$ well-posedness proved by Colombini, De Giorgi and Spagnolo in [3], by Jannelli in [4] and by Nishitani in [5].

2. Statement and Proof of the Result

In this paper it is convenient to set $\langle\xi\rangle=(2+|\xi|^2)^{1/2}$ instead of the usual $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ in order to have log $\langle\xi\rangle$ greater than a positive constant, $\xi\in\mathbf{R}^n$.

Definition 1 *Let p be a function in $C(0,T;S^m)$. We set*

$$\|p^{(\alpha)}_{(\beta)}\|_{LL}=\sup_{\substack{t\in[0,T]\\ x,\xi\in\mathbf{R}^n}}\langle\xi\rangle^{|\alpha|-m}|\partial^\alpha_\xi\partial^\beta_x p(t,x,\xi)|$$

$$+\sup_{\substack{0<|s-t|<1/2\\ x,\xi\in\mathbf{R}^n}}\frac{\langle\xi\rangle^{|\alpha|-m}|\partial^\alpha_\xi\partial^\beta_x p(s,x,\xi)-\partial^\alpha_\xi\partial^\beta_x p(t,x,\xi)|}{|t-s||\log|t-s||}\tag{1}$$

We define the set $LL(0,T;S^m)$ of Log-Lipschitz functions from $[0,T]$ to S^m as the space of functions p such that $\|p^{(\alpha)}_{(\beta)}\|_{LL}<+\infty$ for every α,β.

Let us consider a first order system in $[0,T]\times\mathbf{R}^n$

$$L=\partial_t-K(t,x,D_x)\tag{2}$$

where the matrix $K\in C(0,T;OPS^1)$ is such that

$$L \text{ is strictly hyperbolic}\tag{3}$$

$$K \text{ has principal symbol } K_0\in LL(0,T;OPS^1)\tag{4}$$