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THE GLOBAL SOLUTION OF THE SCALAR NONCONVEX CONSERVATION LAW WITH BOUNDARY CONDITION (CONTINUATION)

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Abstract Using the Kruskov's method[1], we show the uniqueness for the global weak solution of the initial-boundary value problem (1.1)–(1.3) (in the class of bounded and measurable functions).

Key Words Scalar conservation law; boundary condition; nonconvex; uniqueness.
Classification 35L65.

1. Introduction

The initial-boundary value problem for the scalar conservation law was first discussed by Bardos, Leroux and Nedelec[2] (in several space variables). They showed the existence and uniqueness of the global weak solution by vanishing viscosity method and the Kruskov's method. But they had obtained neither the estimation of the solution in the boundary nor the stability in respect to the boundary data. In 1988, Le Floch[3] considered the initial-boundary value problem for the convex conservation law (in the quarter plane). They derived the explicit formula for the exact solution, and proved the uniqueness of the weak solution (in the class of piecewise regular functions). It is well-known that the nonconvex case is more complicated than the convex case, and that the uniqueness in the class of bounded and measurable functions is more perfect than the uniqueness in the class of piecewise regular functions.

We consider the initial-boundary value problem for the nonconvex conservation law (in the quarter plane):

$$\begin{cases} u_t + f(u)_x = 0 & (0 < x < +\infty,\ t > 0) \quad (1.1)\\ u(0,t) = u_b(t) & (t \ge 0) \quad (1.2)\\ u(x,0) = u_0(x) & (0 \le x < +\infty) \quad (1.3)\end{cases}$$

In [4], we first have given the definition of the global weak solution for the problem (1.1)–(1.3). And we have proved the existence of the weak solution for the problem (1.1)–(1.3) by the polygonal approximations method (for $u_0(x), u_b(t)$ are bounded variation functions, $f(u)$ is a locally Lipschitz continuous function). In this paper we first give the estimation of the solution in the boundary. Then we prove the uniqueness in the class of bounded and measurable functions by Kruskov's method[1]. And we obtain the stability in respect to the boundary data and the initial data.

The global weak solution of the initial-boundary value problem for the scalar nonconvex conservation law plays an important role in the mathematical modeling and computations of the one-dimensional sedimentation processes[5,6].

2. Definition of the Weak Solution

Assume that $f(u)$ is a Lipschitz continuous function on $[-M, M], u_0(x), u_b(t)$ are bounded and measurable functions and

$$|f(u) - f(u')| \le L|u - u'|, \quad \forall u, u' \in [-M, M] \tag{2.1}$$

$$-M \le u_0(x), u_b(t) \le M, \quad 0 \le x < +\infty, \ t \ge 0 \tag{2.2}$$

where M, L are arbitrary positive constants.

Definition 2.1[4] *A locally bounded and measurable function $u(x,t)$ on $[0,+\infty) \times [0,+\infty)$ is called a weak solution of the initial-boundary problem (1.1)–(1.3), if for every $k \in R^1$ and for any nonnegative function $\varphi(x,t) \in C_0^\infty([0,+\infty) \times [0,+\infty))$, it satisfies the following inequality:*

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \{|u-k|\varphi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\varphi_x\} dx dt \\ &\quad + \int_0^{+\infty} \operatorname{sgn}(u_b(t)-k)(f(u(0,t))-f(k))\varphi(0,t) dt \\ &\quad + \int_0^{+\infty} |u_0(x)-k|\varphi(x,0) dx \ge 0 \end{aligned} \tag{2.3}$$

where $\operatorname{sgn} x = \begin{cases} 1, & x \ge 0 \\ -1, & x < 0 \end{cases}$.

Lemma 2.1[4] *If $u(x,t)$ is a weak solution of the problem (1.1)–(1.3), then*

$$\operatorname{sgn}(u(0,t)-k)(f(u(0,t))-f(k)) \le 0, \quad \forall k \in I(u(0,t), u_b(t)), \ a.e. \ t \ge 0 \tag{2.4}$$

where $I(a,b) = [\min\{a,b\}, \operatorname{Max}\{a,b\}]$.

We have: