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AN INTEGRABILITY CONDITION FOR MONGE-AMPÈRE EQUATIONS ON A KÄHLER MANIFOLD

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

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Abstract The symmetry group of the Monge-Ampère equation on a Kähler manifold is determined and an integrability condition on the solution is derived as a conservation law.

Key Words Complex Monge-Ampère equation, Futaki's obstruction, Lie symmetry group, Noether's theorem and conservation laws.
Classification 35C, 35J.

0. Introduction

In the study of many geometric differential equations integrability conditions play an important role. For some situations they yield obstructions for solvability, and for the other they provide balancing relation of the problem under consideration among different regions. Notable examples of integrability conditions include Pohozaev's type identities for semi-linear problems in the Euclidean space, the Kazdan-Warner condition for the Nirenberg problem, and the Futaki's obstruction for Kähler-Einstein metrics. It has been known for a long time that all these integrability conditions are intimately related to the symmetry of the underlying manifolds as well as to the canonical differential operators which reflect this symmetry. In [1] we discuss this issue from the viewpoint of Lie's theory of symmetry groups for differential equations and Noether's theorem on conservation laws. The general setting can be briefly described as follows.

Consider a differential operator $\mathcal{F}[u]$ which is the Euler-Lagrange operator for the Lagrangian $\mathcal{L}[u]$ on a manifold with a certain structure. Usually $\mathcal{F}[u]$ is canonical in the sense that it inherits partially or entirely the symmetry of this structure. One would like to look for an integrability condition, or a variational identity as called in [1], for the nonhomogeneous problem

$$\mathcal{F}[u] = g(x, u)$$

The procedure of generating a variational identity corresponding to each (divergence) symmetry of $\mathcal{F}[u] = 0$ consists in the following three steps.

(1) Find the Lie symmetry group for the homogeneous equation

$$\mathcal{F}[u] = 0$$

This can be done by solving a system of linear PDE's satisfied by the infinitesimal generators of the symmetry group.

(2) Determine which infinitesimal symmetry is an infinitesimal variational or divergence symmetry for the Lagrangian $\mathcal{L}$. This step can be easily carried out by direct verification.

(3) Put the infinitesimal divergence symmetry into an expression appearing in a crucial step of the proof of Noether's theorem on conservation laws. After some integration by parts we obtain a variational identity for solutions of the nonhomogeneous problem. In general, each infinitesimal divergence symmetry produces a variational identity.

We have applied this procedure in [1] to derive some variational identities for the p-Laplacian and the conformal Laplacian on a Riemannian manifold. Some of these identities are old, but some are new. In a companion paper [2] we treat the same problem for a class of conformally invariant fourth-order semi-linear equations.

In this note we would like to further illustrate the procedure to the complex Monge-Ampère equation on a compact Kähler manifold. Consider the equation

$$\det\left(g_{\alpha\overline{\beta}} + \frac{\partial^2 u}{\partial z^\alpha \partial \overline{z}^\beta}\right) = \exp F(z, \overline{z}, u) \det\left(g_{\alpha\overline{\beta}}\right) \tag{0.1}$$

which is defined invariantly on a Kähler manifold (M, g). Here $g_{\alpha\overline{\beta}} dz^\alpha \otimes dz^{\overline{\beta}}$ is the Kähler metric in local coordinates and F is a given function on $M \times \mathrm{R}$. Equation (0.1) was studied by Aubin [3] and Yau [4] independently. Both authors proved the existence of a solution when $\partial F/\partial u > 0$. The case $\partial F/\partial u \geq 0$ is much harder and is solved in [4]. Needless to say, the most important case of (0.1) is

$$\det\left(g_{\alpha\overline{\beta}} + \frac{\partial^2 u}{\partial z^\alpha \partial \overline{z}^{\overline{\beta}}}\right) = \exp(\lambda u + \phi(z, \overline{z})) \det\left(g_{\alpha\overline{\beta}}\right) \tag{0.2}$$