J.Partial Diff. Eqs. **11**(1998), 59–72
©International Academic Publishers Vol.11, No.1

APPROXIMATION OF A TWO-PHASE CONTINUOUS CASTING STEFAN PROBLEM

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

(Received May 12, 1997)

Abstract The continuous casting Stefan problem is a mathematical model describing the solidification with convection of a material being cast continuously with a prescribed velocity. We propose a practical piecewise linear finite element scheme motivated by the characteristic finite element method and derive an error estimate for the scheme which is of the same convergence order as that proved for Stefan problem without convection.

Key Words Piecewise linear finite elements, numerical quadrature, error estimates, Stefan problem with convection.

Classification 65N15, 65N30.

1. Introduction

Let Ω be a cylindrical domain $\Omega = \Gamma \times (0, L) \subset \mathbf{R}^d$, $d = 2$ or 3, where $0 < L < +\infty$ and $\Gamma = (0, L_1)$ if $d = 2$ or $\Gamma \subset \mathbf{R}^2$ is a bounded polygonal domain. We write $x = (x', z) \in \Omega$ with $x' \in \Gamma$ and $x_d = z$. Denote by $\Gamma_0 = \Gamma \times \{0\}$, $\Gamma_L = \Gamma \times \{L\}$, $\Gamma_D = \Gamma_0 \cup \Gamma_L$ and $\Gamma_N = \partial\Gamma \times (0, L)$ (cf. Fig.1). For $0 < T < +\infty$, we set $Q_T = \Omega \times (0, T)$.

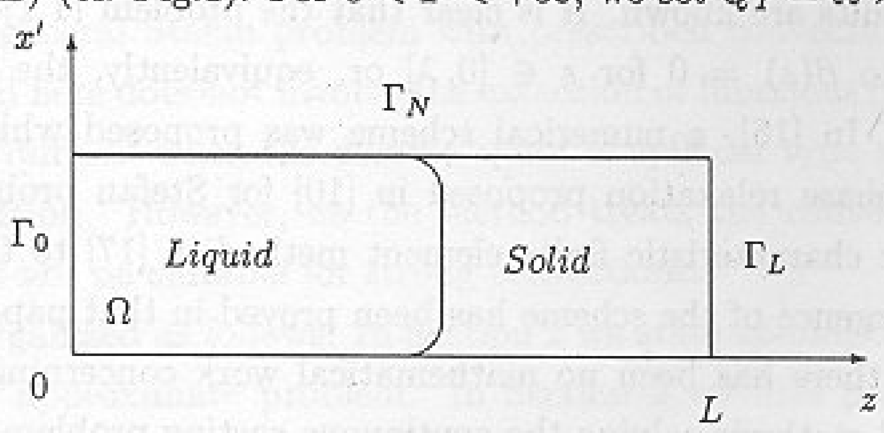

Fig.1 The domain Ω

* This author was partially supported by the National Natural Science Foundation of China.

We consider the following degenerate nonlinear parabolic problem

$$\frac{\partial u}{\partial t} + b(t)\frac{\partial u}{\partial z} - \Delta\theta = 0 \quad \text{in} \quad Q_T \tag{1.1}$$

$$\theta = \beta(u) \quad \text{in} \quad Q_T \tag{1.2}$$

$$\theta = g_D(x,t) \quad \text{on} \quad \Gamma_D \times (0,T) \tag{1.3}$$

$$-\frac{\partial\theta}{\partial \mathbf{n}} = p(x)\theta + g_N(x,t) \quad \text{on} \quad \Gamma_N \times (0,T) \tag{1.4}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega \tag{1.5}$$

where θ stands for the temperature, u is the enthalpy, $b(t) \geq 0$ is the extraction velocity of the ingot, and $\mathbf{n}$ is the unit outer normal to $\partial\Omega$. The mapping $\beta : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous and monotone increasing. It is assumed that $\beta(s) = 0$ for any $s \in [0, \lambda]$ and $0 < \alpha_1 \leq \beta'(s) \leq \alpha_2$ for almost every $s \in \mathbf{R}\backslash[0, \lambda]$, where $\lambda > 0$ is the latent heat. It is clear that the inverse mapping $H = \beta^{-1}$ is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ which is Lipschitz continuous in $\mathbf{R}\backslash 0$ and has a jump discontinuity at 0.

The multidimensional two-phase Stefan problem without convection (i.e. when $b = 0$) has been studied by many authors. For the existence and uniqueness of the weak solutions, we refer to[1] and [2]. The convergence of numerical methods for the enthalpy formulation of Stefan problem has been studied in [3], [4], [5] and [6]. The error analysis of the finite element schemes has also been considered in the literature (cf. e.g. [7], [8], [9], [10], [11] and the references therein). The problem (1.1)–(1.5) models a popular industrial solidification process with convection in which a material is cast continuously with prescribed velocity $\mathbf{v} = b(t)\mathbf{e}_d$, where $\mathbf{e}_d = (0, \cdots, 0, 1) \in \mathbf{R}^d$ (cf. e.g. [12] and [13] for the description of the industrial process and mathematical modelling). The existence and uniqueness of the problem (1.1)–(1.5) has been studied in [14] and [15]. Concerning the numerical solutions of the continuous casting problem, relatively few results are known. It is clear that the problem (1.1)–(1.2) is convection dominated due to $\beta(s) = 0$ for $s \in [0, \lambda]$ or, equivalently, the jump discontinuity of the enthalpy. In [16], a numerical scheme was proposed which is based on the nonequilibrium phase relaxation proposed in [10] for Stefan problem to smooth the enthalpy and the characteristic finite element method in [17] to treat the convection term. The convergence of the scheme has been proved in that paper. However, to our best knowledge, there has been no mathematical work concerning the error analysis for the numerical methods solving the continuous casting problem (1.1)–(1.5). In this paper we will propose and study a new scheme which is motivated by the characteristic finite element method in [16] to treat the convection term.

Denote by $\tau > 0$ the time step and $t^n = n\tau$ for any integer $n \geq 0$. Set $\overline{x} = x - b(t)\tau\mathbf{e}_d$