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# INTERPOLATING BETWEEN LI-YAU'S AND HAMILTON'S HARNACK INEQUALITIES ON A SURFACE*

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**Abstract** We establish a one-parameter family of Harnack inequalities connecting Li and Yau's differential Harnack inequality for the heat equation to Hamilton's Harnack inequality for the Ricci flow on a 2-dimensional manifold with positive scalar curvature.
**Key Words** Harnack inequality; heat equation; parabolic equation; Ricci flow; surface.
**Classification** 58G11, 35K55, 35K10.

In this paper we show that on a closed 2-dimensional Riemannian manifold with positive scalar curvature there is a one-parameter family of differential Harnack inequalities joining the Li-Yau Harnack inequality [1] for solutions of the heat equation to the linear Harnack inequality of Hamilton and the author [2] for the Ricci flow, which extends Hamilton's Harnack inequality [3] for the Ricci flow on surfaces. For some further works on differential Harnack inequalities for parabolic equations arising in differential geometry the reader may consult the papers by Andrews [4], Cao [5], Cao and Yau [6], the author [7], Chu and the author [8, 9], Hamilton [10,11], Hamilton and Yau [12], and Yau [13, 14].

Let $M^2$ denote a closed 2-dimensional manifold, $g(t)$ a time-dependent Riemannian metric on $M$, and $R(t)$ its scalar curvature. We say that $g(t)$ is a solution to the 'scaled' Ricci flow if

$$\frac{\partial}{\partial t} g_{ij} = -2\varepsilon R_{ij} = -\varepsilon R g_{ij} \tag{1}$$

where the only difference between this and the usual Ricci flow is the non-negative constant $\varepsilon$ in front of the right-hand-side. Let $u$ be a positive solution to the linear parabolic equation

$$\frac{\partial}{\partial t} u = \Delta u + \varepsilon R u \tag{2}$$

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where the Laplacian and scalar curvature are with respect to the evolving metric. Observe that the equations (1) and (2) are affine in $\varepsilon$ so that we may consider these equations as affine combinations of the special cases where $\varepsilon = 0$ and $\varepsilon = 1$. Note also that the rates of diffusion in (1) and (2) are different when $\varepsilon \neq 1$. In the case where $\varepsilon = 0$, which is the heat equation, Li and Yau [1] proved a differential Harnack inequality in all dimensions (sharp when $R_{ij} \geq 0$). In the case where $\varepsilon = 1$, Hamilton and the author [2] proved a linear Harnack inequality for the Ricci flow in all dimensions, extending Hamilton's trace Harnack inequality [15] for the Ricci flow (which is a special case of his higher dimensional matrix result). Thus the motivation for considering (1) and (2) is to affinely interpolate between the Harnack inequalities in [1] and [2].

As in the previous works on differential Harnack inequalities, we consider the logarithm of $u$, whose evolution is derived from (2) to be

$$\frac{\partial}{\partial t} \ln u = \Delta \ln u + |\nabla \ln u|^2 + \varepsilon R \tag{3}$$

Define the Harnack quantity, analogous to Li-Yan [1] and Hamilton [3] by

$$Q = \frac{\partial}{\partial t} \ln u - |\nabla \ln u|^2 = \Delta \ln u + \varepsilon R \tag{4}$$

We then have the following Harnack inequality for $u$.

**Theorem 1** *Let $(M^2, g_0)$ be a closed surface with $R(g_0) > 0$ and $\varepsilon \geq 0$ any non-negative constant. Let $g(t)$ be a solution to the scaled Ricci flow (1) with $g(0) = g_0$ and $u(t)$ a solution to (2) with $u(0) > 0$. Then $R(t) > 0$ and $u(t) > 0$ as long as the solution to the scaled Ricci flow exists, and*

$$\frac{\partial}{\partial t} \ln u - |\nabla \ln u|^2 + \frac{1}{t} \geq 0 \tag{5}$$

An immediate consequence is

**Corollary 2** 1. *Taking $\varepsilon = 0$, we have the following special case of Li and Yau's result: if $(M^2, g)$ is a closed surface with $R > 0$, and $u$ is a positive solution to the heat equation, then inequality (5) holds.*

2. *If $\varepsilon = 1$ and $u(0) = R(g_0)$, then $u(t) = R(t)$ for $t \geq 0$, and we obtain Hamilton's trace Harnack inequality for the Ricci flow on surfaces: If $(M^2, g(t))$ is a solution to the Ricci flow on a closed surface with $R(0) > 0$, then $R(t) > 0$ for $t \geq 0$ and*

$$\frac{\partial}{\partial t} \ln R - |\nabla \ln R|^2 + \frac{1}{t} \geq 0$$

**Proof of Theorem 1** The fact that $u(t) > 0$ and $R(t) > 0$ for $t \geq 0$ follows from applying the maximum principle to (2) and (6) below. By using the definition (4), the equation (3), and the standard formulae

$$\frac{\partial}{\partial t} R = \varepsilon(\Delta R + R^2), \qquad \frac{\partial}{\partial t} \Delta = \varepsilon R \Delta \tag{6}$$

a straightforward computation yields that the evolution of $Q$ is given by