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THE GLOBAL CAUCHY PROBLEM FOR THE CRITICAL AND SUBCRITICAL NONLINEAR SCHRÖDINGER EQUATIONS IN H^s *

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Abstract In this paper, the local and global existence of solutions for the Cauchy problem of the critical and subcritical nonlinear Schrödinger equations in H^s $(s \geq 2)$ is studied by using the manner of formal differentiation with respect to time. Some conjectures posed by Cazenave and Weissler in [1] are verified.

Key Words Nonlinear Schrödinger equation; Cauchy problem; critical and subcritical powers in H^s; global existence.
Classification 35Q20, 35L05.

1. Introduction

There are some recent papers [1, 2] devoted to the study of the Cauchy problem for nonlinear Schrödinger equations

$$iu_t + \Delta u - \lambda |u|^p u = 0, \quad u(0,x) = \varphi(x) \tag{1.1}$$

in the critical space H^s $(p = 4/(n-2s), 0 \leq s < n/2)$, where $u(t,x)$ defined in $\mathbf{R} \times \mathbf{R}^n$ is a complex valued function, Δ is the Laplace operator on $\mathbf{R}^n$ and $\lambda \in \mathbf{R}$. It is known that the solutions of (1.1) formally satisfy the conservations of charge

$$\|u(t)\|_{L^2(\mathbf{R}^n)} = \|\varphi\|_{L^2(\mathbf{R}^n)} \tag{1.2}$$

and energy

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbf{R}^n)} + \frac{\lambda}{p+2}\int_{\mathbf{R}^n} |u(t,x)|^{p+2} dx = E(\varphi) \tag{1.3}$$

Let us recall that $p = 4/(n-2s)$ $(s \geq 0)$ is said to be an H^s-critical exponent and $p \in (0, 4/(n-2s))$ $(p \in 0, \infty)$ if $s \geq n/2)$ is called an H^s-subcritical exponent for the

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problem (1.1). A large amount of work has been devoted to the study of the global existence of (1.1) in H^1-subcritical cases (See [3, 4, 5]) and their references). It is known in particular that for initial data in H^1 and H^2, the corresponding solutions are called finite energy strong solutions and strong solutions, respectively. The classical global solutions in H^1-subcritical cases for space dimensions $n \leq 11$ were established by Hayashi and Tsutsumi in [6] and [7]. Also, some nontrivial generalizations to the nonlinearities were considered by Kato in [3] and [4]. Recently, by resorting to the analytical method, Struwe [2] has shown the spherically symmetric solutions in the $H^{1/2}$-critical case. For general H^s-critical cases, Cazenave and Weissler in [1] gave a systematic consideration to the local and global solutions in H^s.

The method of formal differentiation with respect to time for nonlinear Schrödinger equations was first used by Segal in [8]. Kato in [3], Hayashi, M. Tsutsumi in [6] and [7], Y. Tsutsumi in [5], Cazenave and Weissler in [1] have developed this method in the most recent years. Using this method, we can decrease one time of differentiation for the nonlinear term $\lambda|u|^p u$ and avoid the loss of derivatives so that the existence of solutions in H^s covers the sufficiently small p. We remark that for the smaller p, it is more difficult for us to prove the existence of the regularity solutions. Cazenave and Weissler in [1] studied the case $s = 2$ by formally differentiating nonlinear Schrödinger equation with respect to time and proved the local existence of solutions of (1.1) if p is an H^2-critical or H^2-subcritical exponent. Also, they showed that the solutions are global if p is H^2-critical and $\|\varphi\|_{\dot{H}^2}$ is sufficiently small. Moreover, they conjectured their work could most likely be developed for all $s \geq 0$ (See [1], Theorem 1.4). We shall continue the work of Cazenave and Weissler and prove the global existence of solutions of (1.1) in H^s for all $s \geq 2$ if p is an H^s-(sub)critical exponent and $\|\varphi\|_{\dot{H}^{n/2-2/p}}$ is sufficiently small.

We first state our main results. For the sake of convenience, let us follow [1] and give the following

Definition *(q, r) is said to be an admissible pair if $r \in [2, 2n/(n-2))$ and $2/q = n(1/2 - 1/r)$. (If $n = 1, 2$, $2 \leq r < \infty$ is allowed.)*

The following is a particular admissible pair (γ, ρ) defined by

$$\gamma = 2 + p, \quad \rho = \frac{2n(2+p)}{n(2+p) - 4} \tag{1.4}$$

Theorem 1.1 *Let $s = n/2 - 2/p \geq 2$ and $[s]^* \leq p + 1$. Then there exists a solution of (1.1) satisfying $\partial_t^j u \in L^\infty(0, \infty; H^{s-2j}(\mathbf{R}^n))$ $(j = 0, 1, \cdots, [s/2])$ if $\varphi \in H^s$ and $\|\varphi\|_{\dot{H}^s}$ is sufficiently small. Moreover, this solution enjoys the following properties:*

(i) *$\partial_t^j u \in L^q(0, \infty; B^{s-2j}_{r,2})$ $(j = 0, 1, \cdots, [s/2])$ for every admissible pair (q, r).*

* $[s]$ denote the largest integer less than s