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NONLINEAR STABILITY OF SHOCK PROFILES FOR NON-CONVEX MODEL EQUATIONS WITH DEGENERATE SHOCK*

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Abstract This paper is concerned with the stability of shock profiles for one-dimensional non-convex equations of viscous materials. The main purpose is to show that the shock profile solution is stable in an appropriate weighted norm space for the case of the degenerate shock, provided that the shock is weak and the initial disturbance is small and of integral zero. The proof is given by means of an elementary but technical weighted energy method to the integrated system of the original one. Moreover, the stability result can be applied to the equation of van der Waals fluid and viscoelascity.
Key Words Stability; degenerate shock; non-convex nonlinearity.
Classification 35K55, 35B40, 36L65.

1. Introduction

This paper is concerned with the stability of degenerate shock profiles for some model equation with a non-convex constitutive relation.

Historically, the first result on the stability of shock profiles was proved for a single equation with convex nonlinearity

$$u_t + f(u)_x = \mu u_{xx}, \quad x \in \mathbf{R},\ t \geq 0 \tag{1.1}$$

where $\mu > 0$ is a constant and $I(f) = \emptyset$, $I(f)$ denotes the set of inflection point of f. This stability result is due to Il'in and Oleinik [1] and its proof was based on the maximum principle. Subsequently, another proof based on the spectral analysis was given by Sattinger [2]. Kawashima-Matsumura [3] investigated the algebraic time decay rate in L^2-framework for small initial disturbances by an elementary energy method.

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On the other hand, the stability theory of shock profiles in system case began with the independent works of Goodman [4] and Matsumura-Nishihara [5], both of which are due to an elementary energy method. After these two papers, the stability of shock profiles was proved for several interesting systems in viscous gas dynamics [3, 5–7], the simplest example of such systems is the equation of barotropic viscous gas considered in [5]

$$\begin{cases} v_t - u_x = 0 \\ u_t + p(v)_x = \mu\Big(\dfrac{u_x}{v}\Big)_x, \quad x \in \mathbf{R},\ t \geq 0 \end{cases} \tag{1.2}$$

where $\mu > 0$ is a constant, and $p'(v) < 0$ and $p''(v) > 0$ for $v > 0$. The stability proofs in all these papers, however, make use of the convex nonlinearity of the equations (for example, the convexity of $p(v)$ in (1.2)) and hence are not applicable straightforwardly to non-convex nonlinearity case. Recently, we observed that the sign of $\overline{Q}''$ (See (3.28)) play an important role in the stability proofs, and the coupling effect would induce to the non-convexity of $\overline{Q}$ even for the case that $p(v)$ is convex [8].

There has been considerable recent interest in extending the stability theory of shock profiles to "non-convex" systems, i.e., systems not satisfying the genuine nonlinearity condition of Lax. For the scalar equation (1.1) with non-convex f, the stability theorems have been established by Kawashima-Matsumura [9], Jones-Gardner-Kapitula [10], Serre [11] and Liu [12]. Kawashima-Matsumura [9] started with the case

$$uf''(u) > 0 \quad \text{for} \quad u \neq 0 \tag{1.3}$$

by an elementary energy method (their results exclude the degenerate shock case). Jones-Gardner-Kapitula [10] treated much more general case than (1.3) under the Lax's shock condition $f'(u_+) < s < f'(u_-)$ and showed both the stability and the time decay rate by the spectral analysis. Serre [11] obtained the L^1-stability of shock profiles for general non-convex f based on the L^1-contraction of the solution. Just recently, Liu [12] showed the stability and the time decay rate for general non-convex f under the Lax's shock condition. All proofs in [9; 12] are based on the weighted energy method where suitable selection of the weight functions plays an essential role. Recently, Kawashima-Matsumura [9] showed a stability theory of viscous shock profiles for some regularized conservation law systems without genuine nonlinearity first time. A typical example of such systems is the following one of viscoelasticity,

$$\begin{cases} v_t - u_x = 0 \\ u_t - \sigma(v)_x = \mu u_{xx}, \quad x \in \mathbf{R},\ t \geq 0 \end{cases} \tag{1.4}$$

under the conditions that the stress function $\sigma(v)$ satisfies $\sigma'(v) > 0$ for all v and $v\sigma''(v) > 0$ for $v \neq 0$. For such non-convex $\sigma(v)$, the stability of the weak viscous shock