J.Partial Diff. Eqs. 12(1999), 1–10
©International Academic Publishers Vol.12, No.1

$C^{1,\alpha}$ REGULARITY OF VISCOSITY SOLUTIONS OF FULLY NONLINEAR PARABOLIC PDE UNDER NATURAL STRUCTURE CONDITIONS

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(Received Nov. 30, 1994)

Abstract In this paper, we concern the fully nonlinear parabolic equations $u_t + F(x,t,u,Du,D^2u) = 0$. Under the natural structure conditions as that in [1], we obtain the $C^{1,\alpha}$ estimates of the viscosity solutions.

Key Words Fully nonlinear parabolic equations; $C^{1,\alpha}$ regularily; viscosity solutions.

Classification 35K55, 35K60.

In [1], Chen Yazhe established the interior $C^{1,\alpha}$ regularity of viscosity solutions to fully nonlinear elliptic equations under natural structure conditions. We shall extend this result to parabolic equations.

Consider the Dirichlet problem of fully nonlinear parabolic equations

$$u_t + F(x,t,u,Du,D^2u) = 0 \qquad \text{in } Q_T = \Omega \times (0,T] \tag{1}$$

$$u = \varphi(x,t) \qquad \text{on } \partial^* Q_T = (\partial\Omega \times (0,T]) \cup (\Omega \times \{0\}) \tag{2}$$

$F(x,t,r,p,X)$ is a continuous function on $\Omega \times (0,T] \times \boldsymbol{R} \times \boldsymbol{R}^N \times \boldsymbol{M}^N$, where $\boldsymbol{M}^N$ denotes the space of $N \times N$ symmetric matrices equipped with usual order, and satisfies

$$\lambda \operatorname{Tr}(Y) \le F(x,t,r,p,X) - F(x,t,r,p,X+Y) \le \Lambda \operatorname{Tr}(Y) \qquad \forall Y \in \mathrm{M}^N,\ Y \ge 0 \tag{3}$$

$$F(x,t,r,p,X) - F(x,t,s,p,X) \ge 0 \qquad \forall r \ge s. \tag{4}$$

Let's give the definition of viscosity solutions.

Definition *Let u be an upper (resp. lower) semi-continuous function in Q_T. u is said to be a viscosity subsolution (resp. supersolution) of (1) if for all $\varphi(x,t) \in C^{2,1}(Q_T)$ at each local maximum (resp. minimum) point $(x_0,t_0) \in Q_T$ of $u - \varphi$, we have*

$$\varphi_t(x_0,t_0) + F(x_0,t_0,u(x_0,t_0),D\varphi(x_0,t_0),D^2\varphi(x_0,t_0)) \le 0$$

$$(resp.\ \varphi_t(x_0,t_0)+F(x_0,t_0,u(x_0,t_0),D\varphi(x_0,t_0),D^2\varphi(x_0,t_0))\geq 0)$$

$u\in C(Q_T)$ is said to be a viscosity solution of (1) *if it is both a viscosity subsolution and a viscosity supersolution of* (1).

Under natural structure conditions and some property of the boundary, the author in [1] used Perron's method and obtained the existence of viscosity solution for the Dirichlet problem of elliptic equations based on the comparison principle. He also proved the Lipschitz continuity of the viscosity solution. We can obtain these for parabolic equations. The details are described in [2]. In this paper, we shall derive the $C^{1,\alpha}$ estimates for the viscosity solution u of (1)-(2) with $u\in C^1$ under the following additional conditions

$$\begin{aligned}&|F(x,t,r,p,X)-F(y,t,r,q,X)|\\&\quad\leq \mu_1(|r|,|p|)\{1+[\mu(|x-y|)+|p-q|^{1/2}\mu(|p-q|)]\|X\|\}\end{aligned}\tag{5}$$

$$|F(x,t,r,p,0)|\leq \mu_2(|r|,|p|)\tag{6}$$

where $\mu(\cdot)$ is nondecreasing, $\mu(\sigma)\to 0$ as $\sigma\to 0$, $\mu(\sigma)/\sigma\geq 1$ in $(0,+\infty)$ and $\int_{0+}\mu(\sigma)/\sigma d\sigma<\infty$, $\mu_i(s,t)$ is nondecreasing with respect to s and $t(i=1,2)$.

Theorem 1 *Assume that F satisfies* (3)-(6). *Let u be a Lipschitz continuous viscosity solution of* (1). *Then Du is Hölder continuous.*

We require the parabolic analogue of Proposition 3.1 in [1]. It takes the following form ([2]):

Lemma 2 *Let u and v be, respectively, a viscosity subsolution and a viscosity supersolution of* (1). *$\Psi(x,y,t)\in C^{2,1}(\Omega\times\Omega\times(0,T])$. If $u(x,t)-v(y,t)-\Psi(x,y,t)$ attains its maximum in $\Omega\times\Omega\times(0,T]$, then there exists $(\bar{x},\bar{y},\bar{t})\in\Omega\times\Omega\times(0,T]$ and $X,Y\in\mathbf{M}^N$ such that*

$$u(\bar{x},\bar{t})-v(\bar{y},\bar{t})-\Psi(\bar{x},\bar{y},\bar{t})=\max\{u(x,t)-v(y,t)-\Psi(x,y,t)\}\tag{7}$$

$$\Psi_\tau(\bar{x},\bar{y},\bar{t})+F(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\Psi(\bar{x},\bar{y},\bar{t}),X)-F(\bar{y},\bar{t},v(\bar{y},\bar{t}),-D_y\Psi(\bar{x},\bar{y},\bar{t}),-Y)\leq 0\tag{8}$$

and

$$\begin{pmatrix}X&0\\0&Y\end{pmatrix}\leq D^2\Psi(\bar{x},\bar{y},\bar{t})\tag{9}$$

Before proving Theorem 1, we need some results in [3].

Let's introduce the Pucci's extremal operators:

$$\mu^-(D^2u)=\lambda\sum_{e_i>0}e_i+\Lambda\sum_{e_i<0}e_i$$

$$\mu^+(D^2u)=\Lambda\sum_{e_i>0}e_i+\lambda\sum_{e_i<0}e_i$$