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ON THE EXISTENCE AND STABILITY OF POSITIVE SOLUTIONS FOR SOME PAIRS OF DIFFERENTIAL EQUATIONS*

Guo Zongming
(Department of Mathematics, Henan Normal University, Xinxiang 453002, China)

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Abstract In this paper, we are concerned with the existence and stability of the positive solutions of a semilinear elliptic system

$$-\Delta u(x) = a(x)v^{\delta}(x) + e(x)$$
$$-\Delta v(x) = b(x)u^{\mu}(x) + m(x) \quad \text{in } \Omega$$
$$u = v = 0 \quad \text{on } \partial\Omega$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. It is shown that under the suitable conditions on δ, μ, there exist a stable and an unstable positive solutions for this system if e and m are sufficiently small in L^∞.

Key Words Positive solutions; multiple solutions; stability; semilinear differential systems.

Classification 25J35, 35B32.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$ and $e, m \in L^\infty(\Omega)$, $e, m \geq 0$ in Ω and $e, m \not\equiv 0$ in Ω. In this paper we are concerned with the existence and stability for positive solutions of a semilinear differential system

$$-\Delta u(x) = a(x)v^{\delta}(x) + e(x) \tag{1.1}$$
$$-\Delta v(x) = b(x)u^{\mu}(x) + m(x) \quad \text{in} \quad \Omega \tag{1.2}$$
$$u = v = 0 \quad \text{on} \quad \partial\Omega \tag{1.3}$$

where $a, b \in C^0(\overline{\Omega})$ with

$$\tilde{a} := \min_{\overline{\Omega}} a(x) > 0, \quad \tilde{b} := \min_{\overline{\Omega}} b(x) > 0$$

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By a positive solution (u,v) of the problem (1.1)–(1.3) we mean that $(u,v) \in C^2(\Omega) \times C^2(\Omega)$ and $u > 0$, $v > 0$ in Ω.

When $e \equiv m \equiv 0$ in Ω, the existence of at least one positive solution of (1.1)–(1.3) has been studied recently by the author [1] under the suitable assumptions on δ and μ. In this paper, we shall show that under the same assumptions on δ and μ as in [1] and that $\|e\|_{L^\infty}$, $\|m\|_{L^\infty}$ are sufficiently small, the problem (1.1)–(1.3) has at least two positive solutions. Moreover, we also study the stability of such solutions. The existence of positive solutions of such kind of problems has been studied by many authors, see, for example, [2–6]. In all of these papers, the solutions were obtained by means of variational principles. The advantage of such methods is that the systems with some kinds of general nonlinearities can be handled. The shortcomings of such approaches are that they can not be easily used to discuss the systems with more than two equations; and to discuss the stability of solutions, meanwhile can not be easily used to discuss the existence and stability of the solutions of the corresponding parabolic systems. In this paper, we use the degree theory to study the existence of positive solutions of (1.1)–(1.3). Meanwhile, we also deal with the stability of such solutions. We should mention that our methods in this paper can be used to deal with the existence and stability of positive periodic solutions of the corresponding parabolic differential system

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u = a(t,x)v^\delta + e(t,x)\\
&\frac{\partial v}{\partial t} - \Delta v = b(t,x)u^\mu + m(t,x) \quad \text{in} \quad \mathbf{R}_+ \times \Omega\\
&u = v = 0 \quad \text{on} \quad \mathbf{R}_+ \times \partial\Omega\\
&u(t) = u(t+T), \quad v(t) = v(t+T) \quad \text{in} \quad \overline{\Omega}\\
&u > 0, \quad v > 0 \quad \text{on} \quad \mathbf{R}_+ \times \Omega
\end{aligned}
$$

Our methods of this paper can also be used to deal with the systems with more than two differential equations. The systems with more general nonlinearities need further discussion.

2. Existence Results

In this section, we first describe our existence theorem and then use the Leray-Schauder degree to show the existence of positive solutions. Throughout the rest of this paper, we set $E = C^0(\Omega) \times C^0(\Omega)$, denote by$\|\cdot\|_q$ the norm of $L^q(\Omega)$. We denote K the cone of non-negative functions in E.