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TIME-PERIODIC SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS WITH ELLIPTICITY*

Tang Shaoqiang
(Department of Mechanics and Engineering Science, Peking University, Beijing 100871, China)

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Abstract We study the time-periodic solutions to systems of conservation laws with ellipticity. It is shown that under time-periodic boundary condition, the system admits at least one global time-periodic solution bounded uniformly with the same period.

Key Words Time-periodic solutions; ellipticity.
Classification 35H05.

1. Introduction

Time-periodic behaviour is commonly observed in physics and other systems of interest. Elementary methods like Fourier analysis may discover such solutions for linear systems. However, when nonlinearity comes in, difficulties arise due to the lack of rigorous tools. Hence the demands from nonlinear sciences are far from being satisfied.

Time-periodic solutions have been studied by Greenberg and Rascle for a cooked-up system of nonlinear conservation laws [1]. As for physical models, Matsumura and Nishida [2] studied one-dimensional piston problem of viscous ideal gas, and showed if the piston motion is periodic in time, there exists at least one periodic solution with the same period. Luo has then extended the result to polytropic gas [3].

If we ignore the viscosity in aforementioned systems temporarily, the 'principal' part of the system is hyperbolic. Extensive explorations have been made with the initial-boundary value problem, Cauchy problem, outer-pressure problem, etc. of this kind of 'hyperbolic' conservation laws with dissipation, e.g. [4–7]. Experiences have therefore been accumulated. On the other hand, if the 'principal' part is not hyperbolic, the systems may exhibit complicated and exciting phenomena, including phase-transitions, chaos, and time-periodic solutions, e.g. [8, 9]. However, the time-periodic problem has not yet been tackled theoretically.

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In this paper, we would like to investigate the time-periodic solutions to the nonlinear systems of the following form:

$$\begin{cases} u_t - v_x = 0 \\ v_t - (\sigma(u) + \mu v_x)_x = 0 \end{cases} \tag{1.1}$$

in the domain $(x, t) \in [0, 1] \times R$, where μ is a positive constant (viscosity coefficient). If we ignore the dissipative term μv_{xx} in the second equation, the eigenvalues are $\lambda = \pm\sqrt{\sigma'(u)}$. Therefore, the 'principal' part is hyperbolic if $\sigma'(u) > 0$, and elliptic if $\sigma'(u) < 0$. We consider the elliptic case in this paper. We shall show that the system (1.1), under boundary conditions

$$\begin{cases} u(0, t) = 0 \\ u(1, t) = Q(t) \end{cases} \tag{1.2}$$

with $Q(t)$ a given T-periodic C^2 function, admits at least one T-periodic solution that is uniformly bounded.

The main result is stated and proved in Section 2. We also give some further discussions on related topics in Section 3.

2. Main Result and Proof

Noticing the form of system (1.1), without loss of generality, we may assume

$$\int_0^T \int_0^1 v dx dt = 0 \tag{2.1}$$

$$\sigma(0) = 0 \tag{2.2}$$

We denote

$$A_0 = \sigma'(0) < 0 \tag{2.3}$$

$$\eta = |Q(t)| + |Q'(t)| + |Q''(t)| \tag{2.4}$$

From the second equation of (1.1), we can find the compatibility condition for periodic solution as

$$\int_0^T \sigma(Q(t)) dt = 0 \tag{2.5}$$

Our main result can then be stated as the following theorem.

Theorem 2.1 *For any given $\delta > 0$, the system (1.1) under T-periodic boundary condition (1.2) admits at least one T-periodic solution $(u(x,t), v(x,t))$ with $|u(x,t)| \le \delta$, $|v(x,t)| \le \delta$, $(x,t) \in [0,1] \times [0,T]$, provided $\eta \le C(\delta)$, and that the compatibility condition (2.5) holds. Furthermore, we have*

$$\begin{cases} u \in C(0,T,H^1); \quad u_t \in C(0,T,L^2) \cap L^2(0,T,H^1) \\ v \in C(0,T,H^1) \cap L^2(0,T,H^2), \quad v_t \in L^2(0,T,L^2) \end{cases}$$