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MODIFIED TRICOMI PROBLEM FOR A NONLINEAR SYSTEM OF SECOND ORDER EQUATIONS OF MIXED TYPE

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Abstract In this paper a nonlinear system of second order equations of mixed type is considered. The existence of H^1 strong solution for the modified Tricomi problem is proved by the energy integral method and the Leray-Schauder's fixed point principle.

Key Words Nonlinear system of second order equations of mixed type; modified Tricomi problem; energy integral method; Leray-Schauder's fixed point principle.

Classification 35M05.

In domain $\mathcal{D} \subset \mathrm{R}^2$ we consider a nonlinear system of second order equations

$$LU \equiv U_{xx} + K(x,y)U_{yy} + BU_y + EU - \operatorname{grad} F(U) = G \tag{1}$$

where $U = (u_1(x,y), u_2(x,y), \cdots, u_N(x,y))$ and $G = (g_1(x,y), g_2(x,y), \cdots, g_N(x,y))$ are N-dimensional vector functions, and $g_j \in L_2(\mathcal{D})$, $j = 1, \cdots, N$; B and E are $N \times N$ symmetric matrices, whose elements b_{kl} and e_{kl} $(k, l = 1, \cdots, N)$ are functions of x and y, $b_{kl}, e_{kl} \in C^1(\mathcal{D})$, F is a positive nonlinear scalar function of vector U; $K(x,y)$ is a diagonal matrix

$$K(x,y) = \begin{pmatrix} k_1(x,y) & & 0 \\ & \ddots & \\ 0 & & k_N(x,y) \end{pmatrix} \tag{2}$$

where $k_j(x,y)$, $j = 1, \cdots, N$ satisfy the following conditions:

$$\begin{cases} \text{(i) } k_j(x,y) \begin{cases} > 0, & y > 0 \\ = 0, & y = 0, \\ < 0, & y < 0 \end{cases} \quad k_j(x,y) \in C^2(\mathcal{D}),\ j = 1, \cdots, N \\ \text{(ii) } k_j(x,y) \le k_0(x,y) < 0, \quad \forall 1 \le j \le N,\ \forall (x,y) \in \mathcal{D} \cap \{y < 0\} \end{cases} \tag{3}$$

Then, (1) is a nonlinear system of second order equations of mixed type and, is elliptic in $\mathcal{D}^+ = \mathcal{D} \cap \{y > 0\}$ as well as is hyperbolic in $\mathcal{D}^- = \mathcal{D} \cap \{y < 0\}$, $\gamma = \mathcal{D} \cap \{y = 0\}$ is its degenerate line.

We consider the system (1) in such a domain $\mathcal{D}:\Gamma_0$ is the outer boundary of $\mathcal{D}^+$ in the part $\{y>0\}$, which is connected with the x-axis at points P and Q; Γ_+ and Γ_- are the outer boundaries of $\mathcal{D}^-$ in the part $\{y<0\}$, which are two characteristic lines, issuing from the points P and Q respectively and are defined by the following equations

$$\Gamma_+: dy+\sqrt{-k_0}dx=0, \quad \Gamma_-: dy-\sqrt{-k_0}dx=0 \tag{4}$$

Assume that the boudnary curves Γ_0,Γ_+ and Γ_- satisfy the following conditions:

$$n_2\mid_{\Gamma_0}\geq 0, \quad n_2\mid_{\Gamma_+\cup\Gamma_-}<0 \tag{5}$$

where (n_1,n_2) are two components of the unit vector $\vec{n}$ of the outward normal to the boundary curve $\partial\mathcal{D}$, $n_1=dy/ds$, $n_2=-dx/ds$.

Then, under the conditions (3) and (5) we may consider the modified Tricomi problem:

$$U=0 \quad \text{on } \Gamma_0 \tag{6}$$

In the case $N=1$, the nonlinear problem (1) (6) was considered in [1].

Let

$$L_0U\equiv U_{xx}+K(x,y)U_{yy}+BU_y+EU=G_1 \tag{7}$$

Make the double integral

$$J=\iint_{\mathcal{D}} e^{qy}U_y\cdot L_0U\,dxdy \tag{8}$$

where

$$q=\begin{cases}\varepsilon, & y\geq 0\\ \lambda, & y<0\end{cases} \tag{9}$$

here ε is an arbitrarily small positive constant, λ is a sufficiently large positive number. After integration by parts, we get

$$\begin{aligned}J=&\iint_{\mathcal{D}}\Big\{\frac{q}{2}U_x\cdot U_x+\Big(B-\frac{1}{2}K_y-\frac{q}{2}K\Big)U_y\cdot U_y-\frac{1}{2}(E_y+qE)U\cdot U\Big\}e^{qy}dxdy\\&+\int_{\partial\mathcal{D}}\Big\{\frac{1}{2}(KU_y\cdot U_y-U_x\cdot U_x+EU\cdot U)n_2+U_y\cdot U_xn_1\Big\}e^{qy}ds\\=&I_1+I_2\end{aligned} \tag{10}$$

Now we assume that the coefficients of the system (1) satisfy the following conditions:

$$\begin{cases}\text{(i) } B-\frac{1}{2}K_y \text{ is a positive definite matrix in } \{y\geq 0\}\\ \text{(ii) } E_y \text{ is a negative definite matrix in } \{y\geq 0\}\\ \text{(iii) } E \text{ is a negative definite matrix in } \{y<0\}\end{cases} \tag{11}$$