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CONVECTIVE POROUS MEDIUM SYSTEMS WITH NONLINEAR FORCING AT THE BOUNDARY*

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Abstract This paper deals with the global positive solutions of convective porous medium systems with nonlinear forcing at the boundary. Necessary and sufficient conditions on the global existence of all positive solutions are obtained.

Key Words Convective porous medium systems; nonlinear boundary conditions; global solution; blow up; upper and lower solutions methods.
Classification 35K55, 35B40.

1. Introduction

In this paper, we study the existence and nonexistence of global positive solutions of the convective porous medium systems

$$\begin{cases} u_t = (u^m u_x)_x + \dfrac{1}{r}(u^r)_x, & 0 < x < 1,\ t > 0 \\ v_t = (v^n v_x)_x + \dfrac{1}{l}(v^l)_x, & 0 < x < 1,\ t > 0 \\ u^m u_x|_{x=0} = 0, u^m u_x|_{x=1} = u^\alpha v^p|_{x=1}, & t > 0 \\ v^n v_x|_{x=0} = 0, v^n v_x|_{x=1} = u^q v^\beta|_{x=1}, & t > 0 \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & 0 \le x \le 1 \end{cases} \tag{1}$$

where m, n, α and β are all nonnegative constants, r, l, p and q are all positive constants. $u_0(x), v_0(x)$ are positive C^1 functions and satisfy the compatibility conditions at $x = 0, 1$. Our main objectives are to obtain the necessary and sufficient conditions on the global existence of all positive solutions.

From a physical point of view, the differential equations in (1) have been suggested as some models, for example, the unsaturated flow of water through a homogeneous,

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isotropic, rigid porous medium; the compressible flow of gas through a saturated porous medium, the convective medium equation, see [1-3] and the references therein. The nonlinear boundary conditions in (1) prescribed at $x=1$ can be physically interpreted as the nonlinear radiation law, which here is actually an absorption law, see [4, 5].

The motivation of our work comes from the recent studies of [2, 3, 6, 7], where the single convective porous medium equation was discussed.

2. Main Results and Preliminaries

Throughout this paper, we denote $\delta=\min\{\min\limits_{[0,1]} u_0(x), \min\limits_{[0,1]} v_0(x)\}$. Because $u_0(x)$, $v_0(x)$ are positive C^1 functions we have $\delta>0$.

By the results of [8] we know that there exist $T>0$ and a unique noncontinuable classical positive solution $(u(x,t),v(x,t))$ of (1) defined on $[0,1]\times[0,T)$. Moreover, if $T<+\infty$, then $\lim\limits_{t\to T-}\sup(\|u(\cdot,t)\|_\infty+\|v(\cdot,t)\|_\infty)=+\infty$. In this case we call that the solution $(u(x,t),v(x,t))$ of (1) blows up in finite time.

Comparison Principle Suppose that $(\bar{u}(x,t),\bar{v}(x,t))$ $((\underline{u}(x,t),\underline{v}(x,t)))$ is the positive classical upper (lower) solution of (1). If $(\bar{u}(x,0),\bar{v}(x,0))\geq(u_0(x),v_0(x))$ $((\underline{u}(x,0),\underline{v}(x,0))\leq(u_0(x),v_0(x)))$, then the solution $(u(x,t),v(x,t))$ of (1) satisfies

$$(u(x,t),v(x,t))\leq(\bar{u}(x,t),\bar{v}(x,t))((u(x,t),v(x,t))\geq(\underline{u}(x,t),\underline{v}(x,t)))$$

Consequently, $u(x,t)\geq\delta$, $v(x,t)\geq\delta$ since (δ,δ) is a lower solution of (1).

Remark 1 The defination of upper (lower) solution is standard.

Remark 2 Due to the quasi-monotonic increasing of the nonlinear forcing at the boundary and the regularity of solution, the proof of Comparison Principle is standard, see [7] and [9].

Our main results read as follows:

Theorem 1 *Assume $r,l\leq1$, then the solution $(u(x,t),v(x,t))$ of (1) exists globally if and only if $\alpha\leq1$, $\beta\leq1$ and $pq\leq(1-\alpha)(1-\beta)$.*

Theorem 2 *Assume $r\leq1$ and $l>1$.*

(i) *If $n\leq l-1$, then the solution $(u(x,t),v(x,t))$ of (1) exists globally if and only if $\alpha\leq1$, $\beta\leq1$, $\beta+l-1\leq n+1$ and $pq\leq(1-\alpha)(n+1-\beta-(l-1))$;*

(ii) *If $n>l-1$, then the solution $(u(x,t),v(x,t))$ of (1) exists globally if and only if $\alpha\leq1$, $\beta\leq1$ and $pq\leq(1-\alpha)(1-\beta)$.*

Theorem 3 *Assume $r>1$ and $l\leq1$.*

(i) *If $m\leq r-1$, then the solution $(u(x,t),v(x,t))$ of (1) exists globally if and only if $\alpha\leq1$, $\beta\leq1$, $\alpha+r-1\leq m+1$ and $pq\leq(m+1-\alpha-(r-1))(1-\beta)$;*

(ii) *If $m>r-1$, then the solution $(u(x,t),v(x,t))$ of (1) exists globally if and only if $\alpha\leq1$, $\beta\leq1$ and $pq\leq(1-\alpha)(1-\beta)$.*

Theorem 4 *Assume $r>1$ and $l>1$.*