J.Partial Diff. Eqs. **12**(1999), 179–192
©International Academic Publishers Vol.12, No.2

INSTANTANEOUS SHRINKING OF SUPPORTS FOR NONLINEAR REACTION-CONVECTION EQUATIONS*

Su Ning
(Dept. of Applied Math., Tsinghua Univ., Beijing 100084, China
E-mail: nsu@math.tsinghua.edu.cn)
(Received June 16, 1997; revised Jan. 30, 1998)

Abstract Support analysis is performed for solutions of nonlinear reaction-convection equations in which both singular flux and source term are included. The positivity versus instantaneous shrinking for the solutions is determined by the relative strength of the flux and the source, as well as the decay rate at infinity of initial value of solutions. As an application of the analysis, the case of power-type nonlinearities is checked in details.

Key Words Reaction-convection; instantaneous shrinking; propagation property.

Classification 35B30, 35R35.

1. Introduction

This work studies the support properties of the following nonlinear reaction-convection equation

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = f(u), \quad x \in \mathbf{R}, \ t > 0 \tag{1.1}$$

where $\varphi(z), f(z) \in C(\mathbf{R})$. It is well known (cf. [1–5]) that under mild conditions there exist unique (generalized, entropy) solutions for the Cauchy problem of the equation (1.1).

Let's recall some known phenomena. For simplicity, assume $f(z) \equiv 0$. Let $u(x,t)$ be a solution of the equation (1.1), and, in a certain sense,

$$\begin{cases} u(x,0) > 0, & x < 0 \\ u(x,0) = 0, & x > 0 \end{cases} \tag{1.2}$$

It is known (cf. [4-5]) that if $\varphi'(0^+) < +\infty$, then for every $t > 0$ there exists $R(t) < +\infty$ such that

$$u(x,t) = 0 \quad \text{for } x \geq R(t)$$

* The research supported by National Natural Science Foundation of China, and by Scientific Research Fund, Tsinghua University.

In the case $\varphi'(0^+) = +\infty$, however, if

$$\liminf_{x \to -\infty} u(x,0) > 0$$

then irrespective of the condition (1.2),

$$u(x,t) > 0 \quad \text{for all } x \in \mathbf{R},\ t > 0 \tag{1.3}$$

This displays an instantaneous expansion of support for the solution.

It should be pointed out that restriction on the behavior of $u(x,0)$ at $-\infty$ is necessary. In fact, as a part of the results established in this work, we shall show that if $u(x,0) \to 0^+$ as $x \to -\infty$, then (1.3) is valid only for $0 < t < A$ when

$$\lim_{x \to -\infty} \left[\frac{|x|}{\varphi'(u(x,0))}\right] = A > 0$$

But for every $t > A$ there exists $L(t) > -\infty$ such that

$$u(x,t) = 0 \quad \text{for } x < L(t)$$
$$u(x,t) > 0 \quad \text{for } x > L(t)$$

which means that support of the solution shrinks instantaneously in finite time! Specially, $A = +\infty$ implies that the shrinking occurs immediately.

More generally, we shall obtain the following results.

Theorem 1.1 *For every nonnegative solution $u(x,t)$ of (1.1) there always exist instantaneous shrinking time $\tau \in [0, +\infty]$, and (left) interface function $L : (\tau, +\infty) \to (-\infty, +\infty)$, such that when $0 < t < \tau$,*

$$u(x,t) > 0 \quad \text{for all } x \in \mathbf{R}$$

and as $t > \tau$,

$$\begin{cases} u(x,t) = 0, & x < L(t) \\ u(x,t) > 0, & x > L(t) \end{cases}$$

Here $\tau = +\infty$ means that $u(x,t) > 0$ for all $x \in \mathbf{R}$ and $t > 0$.

The instantaneous shrinking time is determined by the behaviors of the flux function $\varphi(z)$ and the source function $f(z)$ near $z = 0$, as well as the decay rate of the initial value $u(x,0)$ as $x \to -\infty$.

Theorem 1.2 *Assume that the solution $u(x,t)$ satisfies*

$$h_1(x) \le u(x,0) \le h_2(x) \quad \text{for } x \ll -1$$

where $h_i(x)$ is nonnegative, non-decreasing C^1-functions satisfying

$$h_i(x) \to 0 \quad \text{as } x \to -\infty, \quad i = 1, 2$$