J.Partial Diff. Eqs. 12(1999), 262–280
©International Academic Publishers Vol.12, No.3

HARMONIC MAPS FROM COMPLETE NONCOMPACT MANIFOLDS

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(Received Dec.20, 1997)

Abstract In this paper we prove some general existence theorems of harmonic maps from complete noncompact manifolds with the positive lower bounds of spectrum into convex balls. We solve the Dirichlet problem in classical domains and some special complete noncompact manifolds for harmonic maps into convex balls. We also study the existence of harmonic maps from some special complete noncompact manifolds into complete manifolds with nonpositive sectional curvature which are not simply connected.

Key Words Harmonic map; classical domain; dirichlet problem; character boundary.

Classification 58E20.

1. Introduction

In this paper we study mainly the existence problem of harmonic maps from complete noncompact Riemannian manifolds into small convex balls. Let (M, g) and (N, h) be two complete Riemannian manifolds of dimension m and n respectively, then a mapping $u: M \to N$ is a harmonic map if and only if it is a classical solution of the Euler-Lagrange equation of the energy functional

$$E(f) = \int_M e(f)dV, \quad f \in C^1(M, N)$$

where $e(f)$ is the energy density and in local coordinates we have

$$e(f) = g^{ij}(x)h_{\alpha\beta}(f(x))\frac{\partial f^\alpha}{\partial x^i}(x) \cdot \frac{\partial f^\beta}{\partial x^j}(x)$$

(As a convention, Latin letters run from 1 to m, greek letters run from 1 to n, and repeated indices are to be summed.) The Euler-Lagrange equation is given by

$$\tau^\alpha(u) = \Delta u^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i} \cdot \frac{\partial u^\alpha}{\partial x^j} = 0$$

where $\tau(u)$ is called the tension field of u.

We need also the following formulation of these equations later: Let $\{e_1, \cdots, e_m\}$, $\{\xi_1, \cdots, \xi_n\}$ be local orthonormal frames for the tangent bundles of neighborhoods of $q \in M$ and $u(q) \in N$, respectively. Then $Du = u_{\alpha i}\xi_\alpha \otimes \theta_i \in \Gamma(u^*(TN) \otimes T^*M)$, where $u_*(e_i) = u_{\alpha i}\xi_\alpha$ and $\{\theta_1, \cdots, \theta_m\}$ is the orthonormal coframe dual to $\{e_1, \cdots, e_m\}$. Let $D^2u = u_{\alpha ij}\xi_\alpha \otimes \theta_i \otimes \theta_j$ denote the covariant differential of Du as a section of $u^*(TN) \otimes T^*M$, it is not hard to see that, in terms of local coordinates,

$$D^2u = \left\{ \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} - \tilde{\Gamma}^k_{ij}(x)\frac{\partial u^\alpha}{\partial x^k} + \Gamma^\alpha_{\beta\gamma}(u(x))\frac{\partial u^\beta}{\partial x^i} \cdot \frac{\partial u^\gamma}{\partial x^j} \right\} \frac{\partial}{\partial u^\alpha} \otimes dx^i \otimes dx^j$$

where $\tilde{\Gamma}^k_{ij}$ and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the metrics on M and N respectively. Then, u is a harmonic map if and only if Trace $(D^2u) = 0$, that is, iff $u_{\alpha ii} = 0$, for all α.

Throughout this paper $\overline{B_\tau(p)}$ shall always denote the closed geodesic ball of radius τ and center at p in a complete, C^∞ Riemannian manifold N. Here $\tau < \min\left\{\frac{\pi}{2k},\right.$ injectivity radius of N at $\left.p\right\}$, where $k^2 \geq 0$ is an upper bound for sectional curvatures of N.

In [1] W-Y. Ding and the author of this paper obtained the following results (Theorem 3.1 in [1]).

Theorem 1 *Let (M, g) be a complete Riemannian manifold with $\lambda(M) > 0$. Let (N, h) be a complete Rimannian manifolds with $K_N \leq 0$. Assume that $\phi \in C^2(M, N)$ satisfies $|\tau(\phi)| \in L^p(M)$ for some $p \geq 2$. Then*

(i) *There exists a harmonic map $u \in C^\infty(M, N)$, which is homotopic to ϕ and satisfies*

$$\int_M d_N(u(x), \phi(x))^p dv < \infty$$

such a harmonic map is unique if N is simply connected.

(ii) *If we assume in addition that $p \geq \frac{m}{2}$, $\mathrm{Ric}(M) \geq -K$ for some constant $k > 0$, and*

$$\inf_{x \in M} \mathrm{Vol}\ (B_1(x)) = a > 0$$

then the harmonic map u satisfies

$$d_N(u(x), \phi(x)) \to 0, \quad as \ \ x \to \infty$$

In Section 3 we shall prove the following theorem which is analogous to the above theorem.

Theorem A *Let (M, g) be a complete noncompact Riemannian manifold with $\lambda(M) > 0$ and $\overline{B_\tau(p)}$ be a closed geodesic convex ball. Assume that $\phi \in C^2(M, \overline{B_\tau(p)})$ satisfies $|\tau(\phi)| \in L^p(M)$ and $e(\phi) \in L^p(M)$ for $p \geq 2$. Then*

(i) *There exists a harmonic map u from M into N which satisfies*

$$\int_M d(u, \phi)^{2p} dM < \infty$$