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THE STRONG SOLUTION OF A CLASS OF GENERALIZED NAVIER-STOKES EQUATIONS*

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Abstract We study initial boundary value (IBV) problem for a class of generalized Navier-Stokes equations in $L^q([0,T); L^p(\Omega))$. Our main tools are regularity of analytic semigroup by Stokes operator and space-time estimates. As an application we can obtain some classical results of the Navier-Stokes equations such as global classical solution of 2-dimensional Navier-Stokes equation etc.

Key Words Admissible triple; generalized Navier-Stokes equations; initial boundary value problem; space-time estimates.
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1. Introduction and Main Results

In this paper we consider the following IBV problem for a class of generalized Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \nabla P = f(u, \nabla u), & (x,t) \in \Omega \times [0,T) \\ \operatorname{div} u(\cdot,t) = 0, & (x,t) \in \Omega \times [0,T) \\ u|_{\partial\Omega} = 0 & \\ u(x,0) = \varphi(x), & x \in \Omega \end{cases} \tag{1.1}$$

where $u = (u_1, \cdots, u_n)$ is a vector value function, $P(x,t)$ is a scalar value function, $\varphi = (\varphi_1, \cdots, \varphi_n)$ is an initial data. Let $f : \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ be nonlinear vector functions, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. For $1 \leq p \leq \infty$, $L^p = L^p(\Omega)$ denote the standard Lebesgue space with norm $\|\cdot\|_p$, $E^p(\Omega) = \{u = (u_1, \cdots, u_n) |\ u_i \in L^p(\Omega)$ and $\operatorname{div} u = 0$ in the sense of distribution $\}$. $L^q([0,T); L^p(\Omega))$ denotes space-time Lebesgue space, $L_T^{p,q} = L^q([0,T); E^p(\Omega))$ is the subspace of $L^q([0,T); (L^p(\Omega))^n)$ with norm

$$\|\cdot\|_{p,q,T} = \Big(\int_0^T \|\cdot\|_p^q dt\Big)^{1/q}$$

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Let $r \geq 1$ and σ, p satisfy $0 \leq \sigma = \left(\frac{1}{r} - \frac{1}{p}\right)\frac{n}{2} < 1$, we define

$$X^r_{p,T} = \left\{u | u \in C_b([0,T); E^p),\ \|u\|_{X^r_{p,T}} = \sup_{0 \leq t < T} t^{\sigma} \|u(t)\|_p < \infty\right\} \tag{1.2}$$

where $\|u\|_p = \sum_{i=1}^{n} \|u_i\|_p$. When $T = \infty$, we usually denote $X^r_p = X^r_{p,\infty}$.

According to Helmholtz decomposition (See [1]) we have

$$(L^p)^n = E^p \oplus G^p \text{ (direct sum)} \tag{1.3}$$

where $G^p = \{\nabla g; g \in W^{1,p}\}$. Let $\mathcal{P}_p$ be the continuous projection from $(L^p)^n$ to E^p associated with this decomposition, and let B_p be the Laplace operator with zero boundary condition. Now we define $A_p = -\mathcal{P}_p \Delta$ with domain $D(A_p) = E^p \cap D(B_p)$. It is easy to verify that when $1 < p < \infty$, A_p generates an analytic semigroup $e^{-A_p t}$ in E_p, and A_p has a bounded inverse, where $D(A_p) = \{u | u \in W^{2,p}_0 \cap E_p\}$. Hence we can define the fractional power $A^{\alpha}_p (\alpha \in \mathbb{R})$ and

$$\|A^{\alpha}_p e^{-A_p t}\| \leq C_{\alpha} t^{-\alpha}, \quad \text{for } \alpha \geq 0,\ t > 0 \tag{1.4}$$

(For detail see [1–4]). Usually we drop the subscript p attached to A and $\mathcal{P}$.

This paper is devoted to establish well-posedness theory of (1.1) in $L^{p,q}_T$ and $X^r_{p,T}$. As an application we can obtain well known classical results of the classical Navier-Stokes equations. In this problem the function P is automatically determined (up to a function of t) if u is a known vector function, indeed, $\partial P = (I - \mathcal{P}) f(u, \nabla u)$, where $\mathcal{P}$ is the orthogonal projection of $(L^p)^n$ into E^p. For this reason it suffices to consider u only when we talk about the solution of (1.1).

For the sake of convenience we first introduce some notations. $\nabla = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$, $\nabla_j = \frac{\partial}{\partial x_j}$, $(\cdot, \cdot)$ denotes usual L^2 inner product with respective to space variable.

Definition 1.1 *Let $q > r > 1$, $p \geq r$, we call (p, q, r) as admissible triple if*

$$\frac{1}{q} = \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right)$$

As is a standard practice, applying $\mathcal{P}$ to (1.1), we have

$$\frac{du}{dt} + A_p u = F(u, \nabla u), \quad t > 0; \quad u(0) = \varphi(x) \tag{1.5}$$

where $F(u, \nabla u) = \mathcal{P} f(u, \nabla u)$. Hence we study (1.1) *via* the corresponding integral equation

$$u(t) = e^{-At}\varphi(x) + \int_0^t e^{-A(t-s)} F(u, \nabla u) ds \tag{1.6}$$

in $L^{p,q}_T$ and $X^r_{p,T}$.