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ON POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH AN INDEFINITE NONLINEARITY VIA BIFURCATION*

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Abstract Existence of a positive solution was establsihed via bifurcation theory for semilinear elliptic boundary value problems. With the aid of maximum principles and *a priori* estimates, global behavior of the bifurcation curve was obtained. In particular, a general sufficient condition was given for existence of multiple positive solutions when the nonlinearity is sub-critical and indefinite.
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1. Introduction

Let $n \geq 3$, $\Omega \subset \boldsymbol{R}^n$ be a bounded smooth domain, and take

$$f(t) \in C^1(\boldsymbol{R}), \quad a(x) \in C^1(\bar{\Omega})$$

Consider the elliptic problem

$$-\Delta u = \lambda u + a(x) f(u), \quad x \in \Omega \tag{I}$$

where $\lambda \in \boldsymbol{R}$ is a parameter, together with homogeneous Dirichlet boundary data

$$u(x) = 0, \quad x \in \partial\Omega$$

We say that $(\lambda, u(x))$ is a positive solution of (I) if $u(x) > 0$ for $x \in \Omega$.

The problem is said to have an indefinite nonlinearity when f is nonlinear in u and the function $a(x)$ meets the sign changing condition

$$\Omega^+ = \{x \in \Omega | a(x) > 0\} \neq \emptyset, \quad \Omega^- = \{x \in \Omega | a(x) < 0\} \neq \emptyset$$

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The Equation (I) with an indefinite nonlinearity has been studied by several authors, see for example [1–4] and references therein. Various sufficient and necessary conditions for the existence of positive solutions, as well as multiplicity results, have been established. For instance, using a variational approach, the authors [1] showed the existence of a positive solution of (I) when f has a precise (subcritical) power-like growth at infinity and the function a has a 'thick' zero set (see [1]), among other conditions. Under similar conditions as in [1], the author [4] proved that (I) has infinitely many solutions. In [2], under suitable assumptions, the authors obtained the existence of a positive solution of (I) by combining degree theory with *a priori* estimates and by using a super-subsolution argument.

In this paper we shall treat positive solutions of (I) by combining a (global) bifurcation analysis with *a priori* estimates for positive solutions (a local bifurcation analysis was used in [3]).

Throughout the paper, we shall assume that $f(t)$ is superlinear near the origin, namely,

(f1) $f \in C^1(R)$ with $f'(0) = f(0) = 0$.

Theorem 1.1 *Let $\Omega \subset R^n$ be smooth and bounded and let λ_1 be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Then for any $\gamma > 0$, the equation (I) has a positive solution $(\lambda, u(x))$ such that either*

$$\|u\|_{L^\infty(\Omega)} = \gamma$$

or

$$|\lambda - \lambda_1| = \gamma$$

The existence of a (local) branch of positive solutions near $(\lambda_1, 0)$ is established by using the Crandall-Rabinowitz local bifurcation theory from a simple eigenvalue. With the aid of global bifurcation and maximum principle, one then can show that the (local) positive branch actually must extend to infinity in the (λ, u) plane.

Denote by

$$l = \frac{n+2}{n-2}$$

the Sobolev exponent for R^n. The following growth condition on $f(t)$ at infinity will be used.

(f2) We say that $f(t)$ has subcritical growth at infinity if there exists $p \in (1, l)$ such that

$$\lim_{t\to+\infty} \frac{f(t)}{t^p} = L > 0$$

When f has subcritical growth at infinity, *a priori* estimates for positive solutions of (I) are available. In order to state our next two results, we introduce the sign changing condition on the function $a(x)$.

(A) We say that (I) is indefinite if the function $a(x)$ changes sign on Ω, that is,

$$\Omega^+ = \{x \in \Omega | a(x) > 0\} \neq \emptyset, \quad \Omega^- = \{x \in \Omega | a(x) < 0\} \neq \emptyset$$