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ANISOTROPIC PARABOLIC EQUATIONS WITH MEASURE DATA*

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Abstract In this paper, we prove the existence of solutions to anisotropic parabolic equations with right hand side term in the bounded Radon measure $M(Q)$ and the initial condition in $M(\Omega)$ or in L^m space (with m "small").

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1. Introduction and Statement of Results

The existence of solutions to nonlinear elliptic equations and parabolic equations with measure data has been discussed in [1]–[4]. For the case of anisotropic elliptic equations, L.Boccardo, T.Gallouët and P.Marcellini studied it in [5]. In this paper, we will extend the analogous results of [5] for anisotropic elliptic equations to anisotropic parabolic equations and obtain the appropriate function space for solutions. We will consider the following anisotropic parabolic equations:

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(a(x,t,u,Du)\right) = f & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(x,0) = u_0 & \text{in } \Omega \end{cases}$$

Here Ω is a bounded open set in $R^N, N \geq 2$, with smooth boundary $\partial\Omega, Q$ is the cylinder $\Omega \times (0,T)$, where T is a real positive number, and Σ is the "latreal surface" $\partial\Omega \times (0,T), p_i > 1, i = 1, 2, \cdots, N$.

Let a be a Carathéodory function in $Q \times R \times R^N$. We assume there exist two real positive constants α, β and a nonnegative function $h \in L^1(Q)$, such that for every component a_i of $\mathbf{a}$, almost every $(x,t) \in Q$, and for any $s \in R, \xi \in R^N, \eta \in R^N$,

$$\mathbf{a}(x,t,s,\xi)\xi \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i} \tag{1.1}$$

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$$|a_i(x,t,s,\xi)| \le \beta\left(h(x,t) + |s|^{\bar p} + \sum_{j=1}^{N}|\xi_j|^{p_j}\right)^{1-\frac{1}{p_i}}, \quad i=1,2,\cdots,N \tag{1.2}$$

where $\bar p$ satisfies $\frac{1}{\bar p} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$.

$$[\mathbf{a}(x,t,s,\xi) - \mathbf{a}(x,t,s,\eta)][\xi-\eta] > 0, \quad \xi \ne \eta \tag{1.3}$$

In particular, if $\mathbf{a}$ doesn't depend on x, t and s, namely $\mathbf{a}(x,t,s,\xi) \equiv \mathbf{a}(\xi)$, $\mathbf{a}(\xi)$ is the vector field whose components are $a_i(\xi) = |\xi_i|^{p_i-2}\xi_i, i=1,2,\cdots,N, p_i>1$.

We will specify in the statement of the theorems the different hypotheses on f and u_0. The general case is when f and u_0 are the bounded Radon measures on Q and Ω respectively, we will also consider the more regular case when f and u_0 belong to some Lebesgue or Orlicz space.

Definition 1.1 We will say that u is a solution of (P) if $u \in L^1(0,T;W_0^{1,1}(\Omega))$, $a(x,t,u,Du) \in L^1(Q)$ and u satisfies the equation (P) in the following weak sense:

$$-\int_Q u\phi' dxdt + \int_Q \mathbf{a}(x,t,u,Du)D\phi dxdt = \int_Q \phi df + \int_\Omega \phi(x,0)du_0 \tag{1.4}$$

for every $\phi \in C^\infty(\bar Q)$ which is zero in a neighborhood of $\Sigma \cup (\Omega \times \{T\})$

Set

$$W^{1,(p_i)}(\Omega) = \{u | u \in L^{p_i}(\Omega), D_i u \in L^{p_i}(\Omega)\}, \quad i=1,2,\cdots,N \tag{1.5}$$

Define

$$\|u\|_{W^{1,(p_i)}(\Omega)} = \|D_i u\|_{L^{p_i}(\Omega)} + \|u\|_{L^{p_i}(\Omega)}, \quad \forall u \in W^{1,(p_i)}(\Omega) \tag{1.6}$$

$W^{1,(p_i)}(\Omega)$ becomes reflexive Banach space. We will denote by $W_0^{1,(p_i)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ relative to the norm (1.6) in $W^{1,(p_i)}(\Omega)$. Suppose

$$2 - \frac{1}{N+1} < p_i < \frac{\bar p(N+1)}{N}, \quad i=1,2,\cdots,N \tag{1.7}$$

We now state the main results of this paper.

Theorem 1.1 *Assume (1.1)–(1.3) and (1.7) hold, let* $\bar p \le N + \frac{N}{N+1}$,

$$f \in M(Q), \quad u_0 \in M(\Omega) \tag{1.8}$$

where $M(Q)$ and $M(\Omega)$ denote the space of bounded (finite) Radon measure on Q and Ω respectively.

Then there exists a solution u of the problem (P) such that

$$u \in \bigcap_{i=1}^{N} L^{q_i}(0,T;W_0^{1,(q_i)}(\Omega)), \quad \forall q_i \in \left[1, \frac{p_i}{\bar p}\left(\bar p - \frac{N}{N+1}\right)\right), \quad i=1,2,\cdots,N \tag{1.9}$$