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# TIME-ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR GENERAL NAVIER-STOKES EQUATIONS IN EVEN SPACE-DIMENSION*

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**Abstract** We study the time-asymptotic behavior of solutions to general Navier-Stokes equations in even and higher than two space-dimensions. Through the pointwise estimates of the Green function of the linearized system, we obtain explicit expressions of the time-asymptotic behavior of the solutions. The result coincides with weak Huygan's principle.

**Key Words** Compressible flow; conservation laws; general Navier-Stokes equation; space-dimension; Green's function; time-asymtotic behavior.
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## 1. Introduction

In this paper, we derive a detailed description of the asymptotic behavior of solutions of Cauchy problem for the general Navier-Stokes systems of conservation laws in $n$-dimension, where $n > 2$ is even. General Navier-Stokes equation is

$$\begin{cases} \rho_t + \mathrm{div}\rho v = 0 \\ (\rho v^j)_t + \mathrm{div}(\rho v^j v) + P(\rho, e)_{x_j} = \varepsilon \triangle v^j + \eta \mathrm{div} v_{x_j}, \quad j = 1, \cdots, n \\ (\rho E)_t + (\rho E v + P(\rho, e) v) = \triangle\left(k\left(T(e) + \dfrac{1}{2}\varepsilon|v|^2\right)\right. \\ \qquad +\varepsilon \mathrm{div}((\nabla v)v) + (\eta - \varepsilon)\mathrm{div}((\mathrm{div} v)v) \end{cases} \tag{1.1}$$

Here $\rho(x,t), v(x,t), e(x,t), P = P(\rho, e)$ and $T(e)$ represent respectively the fluid density, velocity, specific internal energy, pressure and normalized tempreture, and $E = e + \frac{1}{2}|v|^2$ is the specific total energy. $k > 0$ is the heat conductivity, $\varepsilon > 0$ and $\eta \geq 0$ are viscosity constants, and div and $\triangle$ are the usual spatial divergence and Laplace operator. We assume throughout that $P(\rho, e)$ and $T(e)$ are smooth in a neighborhood of constant state $(\rho^*, e^*)$ and $P_\rho = P_\rho(\rho^*, e^*) > 0, P_e = P_e(\rho^*, e^*) > 0, p = P(\rho^*, e^*)$ and $d^2 = kT'(\rho^*) > 0$.

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For the equation (1.1) Liu and Zeng (see [1]) studied general hyperbolic-parabolic systems in one-dimension and obtained pointwise estimate. In several space variables, the asymptotic behavior of the solution of the Cauchy problem for Navier-Stokes equations has been studied in [2] and [3] but only in $L^p$ space. As for pointwise estimate, for isentropic Navier-Stokes equations in odd space dimension Liu and Wang gave a pointwise estimate in [4], and Xu gave a pointwise estimate for linearized system in even space-dimension in [5].

The plan of this paper is as follows. The linearized system of (1.1) around the constant state $(\rho^*, v^*, e^*)^T = (1, 0, e^*)^T, (e^* > 0)$ is

$$\begin{cases} \rho_t + \text{div} v = 0 \\ v_t + p_\rho \nabla \rho + p_e \nabla e = \varepsilon \Delta v + \eta \nabla \text{div} v \\ e_t + p \text{div} v = d^2 \Delta e \end{cases} \tag{1.2}$$

We first get the pointwise estimate of Green function $G$ of (1.2), then get the asymptotic behavior of the solution of (1.1) by using Duhamel's principle. Comparing with [5], our main difficulty is that we can't get the explit expression for $G$. So in Section 2 we will introduce a method which allows us to get the estimate without explict representation of the matrix.

In this article, $n > 2$ is space dimension, which is even. $C, \varepsilon$ are positive constants.

## 2. Pointwise Estimate of Green Function

The Green matrix $G$ is defined as the solution of the following problem:

$$\begin{cases} (\partial_t + A_1(D_x) + B_1(D_x))G(x,t) = 0 \\ G(x, 0) = \delta(x) I \end{cases} \tag{2.1}$$

The symbols of $A_1(D_x)$ and $B_1(D_x)$ are $\sqrt{-1}A(\xi)$ and $|\xi|B(\xi)$ respectively, where

$$A(\xi) = \begin{pmatrix} 0 & \xi^T & 0 \\ P_\rho \xi & 0 & P_e \xi \\ 0 & P\xi & 0 \end{pmatrix}, \quad B(\xi) = |\xi|^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon|\xi|^2 I + \eta \xi \xi^T & 0 \\ 0 & 0 & d^2|\xi|^2 \end{pmatrix}$$

$\xi = (\xi_1, \cdots, \xi_n)^T$, and $\delta(x)$ is the Dirac function and I the $(n+2) \times (n+2)$ identity matrix. We apply Fourier transformation to the $x$ variables and get

$$\begin{cases} \hat{G}_t(\xi, t) = -\sqrt{-1} E(\xi) \hat{G}(\xi, t) \\ \hat{G}(\xi, 0) = I \end{cases} \tag{2.2}$$

where $E(\xi) = A(\xi) - \sqrt{-1}|\xi| B(\xi)$.

Let $E_{\alpha,\beta}(\xi) = \beta A(\xi) - \sqrt{-1}\alpha B(\xi)$. From simple calculation we know that it has four diffrent eigenvalues. Arrange them as $\lambda_j^{\alpha,\beta} (j = 1, 2, 3, 4)$. They have multiplicity 1,1,1 and $n-1$ respectively. Let the left and right eigenvectors associated with $\lambda_j^{\alpha,\beta}$