J. Partial Diff. Eqs. **14**(2001), 87–96
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Vol.14, No.1

ASYMPTOTICS OF THE MODULE OF MINIMIZERS TO A GINZBURG-LANDAU TYPE FUNCTIONAL

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(Received Apr. 9, 2000)

Abstract The author proves that the module of minimizers for a Ginzburg-Landau type functional converges to 1. And the estimates on the convergent rate are also presented.
Key Words Ginzburg-Landau type functional; module of the minimizers; the rate of convergence.
1991 MR Subject Classification 35J70.
Chinese Library Classification O175.2.

1. Introduction

Let $G \subset R^n (n \geq 2)$ be a bounded and simply connected domain with smooth boundary ∂G. g be a smooth map from ∂G into S^{n-1} satisfying $W_g^{1,p}(G, S^{n-1}) \neq \emptyset$, where $W_g^{1,p}(G, S^{n-1}) = \{v \in W^{1,p}(G, S^{n-1}); v|_{\partial G} = g\}$. Consider the Ginzburg-Landau-type functional

$$E_\varepsilon(u, G) = \frac{1}{p} \int_G |\nabla u|^p + \frac{1}{4\varepsilon^p} \int_G (1 - |u|^2)^2, \quad p \geq 2$$

which has been well-studied in [1,2] for $p = n = 2$. For other related papers, we refer to [3–5].

The functional of the form $E_\varepsilon(u, G)$ was introduced in the study of superconductivity. Similar models are also used in superfluids and XY-magnetism. The minimizer u_ε of $E_\varepsilon(u, G)$ represents a complex order parameter and $|u_\varepsilon|$ has physics senses, for example, in superconductivity, $|u_\varepsilon|^2$ is proportional to the density of supercoducting electrons (i.e., $|u_\varepsilon| = 1$ corresponds to the superconducting state and $|u_\varepsilon| = 0$ corresponds to the normal state). In superfluids, $|u_\varepsilon|^2$ is proportional to the density of superfluid. Thus it is interesting to study the asymptotic behavior of $|u_\varepsilon|$ as $\varepsilon \to 0$.

Clearly the functional $E_\varepsilon(u, G)$ achieves its minimum on $W = \{v \in W^{1,p}(G, R^n); v|_{\partial G} = g\}$ by a function u_ε and there exists a subsequence u_{ε_k} of u_ε such that

$$\lim_{\varepsilon_k \to 0} u_{\varepsilon_k} = u_p, \quad \text{in } W^{1,p}(G, R^n) \tag{1.1}$$

where u_p is a map of least p-energy with boundary value g. It is not difficult to prove that the minimizers u_ε solve the following Euler equation

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \frac{1}{\varepsilon^p}u(1-|u|^2) \tag{1.2}$$

in the weak sense, and they also satisfy the maximum principle: $|u_\varepsilon| \le 1$ a.e. on G.

The general minimizers and one class of them which is named the regularizable minimizers, will be both concerned with in this paper. It is not obvious that $|u_\varepsilon|$, the module of the minimizer of $E_\varepsilon(u,G)$, converges to 1 in $C_{\mathrm{loc}}(G,R^n)$ when $p=n$, which is clear as $p>n$ because of (1.1) and the embedding inequality. We shall assert it in Section 2. In the case $p>n$, the rate of convergence for $\nabla|u_\varepsilon|$ will be given in Section 3. Section 4, we shall introduce the regularizable minimizers $\tilde{u}_\varepsilon$. The estimates of their convergent rate which are better than that of general minimizers will be presented in Section 5.

2. C_{loc} Convergence for $|u_\varepsilon|$

From (1.1) and the embedding theorem we can say there exists a subsequence u_{ε_k} of u_ε such that $\lim_{k\to\infty}|u_{\varepsilon_k}|=1$ in $C(\bar{G},R^n)$ when $p>n$. Since the limit 1 is unique, we obtain

$$\lim_{\varepsilon\to 0}|u_\varepsilon| = 1, \quad \text{in } C(\bar{G},R^n) \tag{2.1}$$

We always assume $p=n$ in this section. We shall prove the weaker conclusion in this case:

Theorem 2.1

$$\lim_{\varepsilon\to 0}|u_\varepsilon| = 1, \quad \text{in } C_{\mathrm{loc}}(G,R^n).$$

For this purpose, we prove the following proposition at first.

Proposition 2.2 *Assume $u\in W$ is a weak solution of (1.2). For any $\rho>0$, denote $G^{\varepsilon\rho}=\{x\in G;\mathrm{dist}(x,\partial G)>\varepsilon\rho\}$, then there exists a constant $C=C(\rho)$ independent of ε such that*

$$\|\nabla u\|_{L^\infty B(x,\varepsilon\rho/8)} \le C\varepsilon^{-1}, \quad x\in G^{\varepsilon\rho} \tag{2.2}$$

Proof Let $y=x\varepsilon^{-1}$ in (1.2) and denote $v(y)=u(x), G_\varepsilon=\{y=x\varepsilon^{-1};x\in G\}, G^\rho=\{y\in G_\varepsilon,\mathrm{dist}(y,\partial G_\varepsilon)>\rho\}$. Since u is a weak solution, we have

$$\int_{G_\varepsilon}|\nabla v|^{p-2}\nabla v\nabla\phi = \int_{G_\varepsilon}v(1-|v|^2)\phi, \quad \phi\in W_0^{1,p}(G_\varepsilon,R^n)$$

Taking $\phi=v\zeta^p, \zeta\in C_0^\infty(G_\varepsilon,R)$, we obtain

$$\int_{G_\varepsilon}|\nabla v|^p\zeta^p \le p\int_{G_\varepsilon}|\nabla v|^{p-1}\zeta^{p-1}|\nabla\zeta||v| + \int_{G_\varepsilon}|v|^2(1-|v|^2)\zeta^p$$