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BLOW UP IN A SEMILINEAR WAVE EQUATION

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Abstract We consider a semilinear wave equation of the form

$$u_{tt}(x,t) - \Delta u(x,t) = -m(x,t)u_t(x,t) + \nabla\phi(x)\cdot\nabla u(x,t) + b(x)|u(x,t)|^{p-2}u(x,t)$$

where $p > 2$. We show, under suitable conditions on m, ϕ, b, that weak solutions break down in finite time if the initial energy is negative. This result improves an earlier one by the author [1].

Key Words Wave equation; weak solutions; negative energy; blow up; finite time.

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1. Introduction

In [1], we considered the following problem

$$\begin{aligned} &u_{tt}(x,t) - \Delta u(x,t) = -au_t(x,t) + \nabla\phi(x)\cdot\nabla u(x,t) + f(u(x,t)),\ x\in\Omega, t>0\\ &u(x,t) = 0,\ x\in\partial\Omega, t>0\\ &u(x,0) = u_0(x), u_t(x,0) = u_1(x),\ x\in\Omega \end{aligned} \tag{1.1}$$

where a is a strictly positive constant, $\phi \in W^{1,\infty}(\Omega)$, and Ω is a bounded open domain of $\boldsymbol{R}^n$ with a smooth boundary $\partial\Omega$. We showed under the assumptions

$$E_0 := \frac{1}{2}\int_\Omega e^{\phi(x)}[u_1^2(x) + |\nabla u_0(x)|^2] - 2F(u_0(x))]dx < 0 \tag{1.2}$$

$$\int_\Omega e^{\phi(x)}u_0(x)u_1(x)dx \geq \frac{a}{2\gamma}\int_\Omega e^{\phi(x)}u_0^2(x)dx \tag{1.3}$$

that weak solutions blow up in finite time. To prove this result, we exploited a lemma by Kalantarov and Ladyzhenskaya [2],which generalizes relatively the convexity method of Levine [3]. Our goal in this work is to remove assumption (1.3) and to allow the coefficient of the damping factor to be variable. To overcome the difficulty that arises, we adopt the method of Georgiev and Todorova [4], which was mainly designed for the nonlinearly damped cases. Therefore, in the present paper we are concerned with an initial-boundary value problem of the form

$$u_{tt}(x,t) - \Delta u(x,t) = -m(x,t)u_t(x,t) + \nabla\phi(x)\cdot\nabla u(x,t) + b(x)u(x,t)|u(x,t)|^{p-2},\ x \in \Omega, t > 0 \tag{1.4}$$

$$u(x,t) = 0,\ x \in \partial\Omega, t > 0 \tag{1.5}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x),\ x \in \Omega \tag{1.6}$$

where Ω is a bounded open domain of R^n with a smooth boundary $\partial\Omega$, and proves a blow up result without the condition (1.3). ($f(u)$ is taken to be equal to $b(x)u|u|^{p-1}$ only for simplicity).

It is also worth mentionning that the results related to existence, nonexistence, and stability of solutions to damped wave equations have been established by many authors, see references [5–12] below.

To make this paper relatively complete we state, without proof, a local existence theorem. The proof is standard and one can easily adopt, for instance, the proof of [4] or [13]. We first make the following hypotheses

H1) $b \in L^\infty(\Omega), \phi \in W^{1,\infty}(\Omega)$, and $m \in L^\infty(\Omega \times [0,\infty))$ such that

$$b(x) > 0, m(x,t) \geq 0, \forall x \in \Omega, t \geq 0$$

H2) $p > 2$ satisfying $p \leq 2(n-1)/n-2$ if $n \geq 3$.

Theorem 1.1 *Assume that* (H1) *and* (H2) *hold. Then for any initial data*

$$u_0 \in H_0^1(\Omega),\quad u_1 \in L^2(\Omega) \tag{1.7}$$

the problem (1.4)–(1.6) *has a unique weak solution defined on* $[0,T), T$ *is small.*

Remark 1.1 By investigating the proof in [13] carefully, one can easily deduce that theorem 1.1 still holds if $m \equiv 0$ in (1.4).

Remark 1.2 By a weak solution, we mean a function

$$u \in C([0,T); H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega)) \tag{1.8}$$

satisfying the equation (1.4) in the following sense: for each θ in $H_0^1(\Omega)$ we have

$$\frac{d}{dt}\int_\Omega u_t(x,t)\theta(x)dx + \int_\Omega \nabla u(x,t)\cdot\nabla\theta(x)dx + \int_\Omega m(x,t)u_t(x,t)\theta(x)dx$$
$$= \int_\Omega \nabla\phi(x)\cdot\nabla u(x,t)\theta(x)dx + \int_\Omega b(x)u(x,t)|u(x,t)|^{p-2}\theta(x)dx$$

for almost each t in $[0,T)$.

Remark 1.3 This result also holds if $\phi \in L^\infty(\Omega)$ with $\nabla\phi$ defined almost everywhere. In this case u satisfies (1.4) in the following sense: for each θ in $H_0^1(\Omega)$

$$\frac{d}{dt}\int_\Omega e^{\phi(x)}u_t(x,t)\theta(x)dx + \int_\Omega e^{\phi(x)}\nabla u(x,t)\cdot\nabla\theta(x)dx$$
$$= -\int_\Omega e^{\phi(x)}m(x,t)u_t(x,t)\theta(x)dx + \int_\Omega e^{\phi(x)}b(x)|u|^{p-2}u(x,t)\theta(x)dx$$