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GENERALIZED SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM FOR PARABOLIC MONGE-AMPÈRE EQUATION*

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Abstract The existence and uniqueness of generalized solution to the first boundary value problem for parabolic Monge-Ampère equation $-u_t \det D_x^2 u = f$ in $Q = \Omega \times (0, T], u = \varphi$ on $\partial_p Q$ are proved if there exists a strict generalized supersolution u_φ, where $\Omega \subset R^n$ is a bounded convex set, f is a nonnegative bounded measurable function defined on $Q, \varphi \in C(\partial_p Q), \varphi(x, 0)$ is a convex function in $\bar{\Omega}, \forall x_0 \in \partial\Omega, \varphi(x_0, t) \in C^\alpha([0, T])$.
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1. Introduction

In recent years the Monge-Ampère equation, a classical equation from geometry and physics, has received a lot of attention for its role in several new areas of applied mathematics. It is almost impossible to apply classical theory of PDE in this kind of equation because of its nonlinearity and degeneracy. In 1958, the introduction of generalized solution and the proof of its existence and uniqueness given by A.D. Aleksandrov, [1], open a new road to the investigation of Monge-Ampère equation. After that, many people took part in the study of Monge-Ampère equation and arrived at many good results.

$-u_t \det D_x^2 u = f$ is one of the three parabolic analogues to Monge-Ampère equation introduced by N.V. Krylov, in [2]. One of its applications is in the proof of Aleksandrov-Bakel'man type maximum principle for second order parabolic equation. It is needed to investigate this kind of equation thoroughly. In this paper, we consider the following first boundary value problem for this parabolic Monge-Ampère equation

$$-u_t \det D_x^2 u = f \text{ in } Q \tag{1.1}$$

$$u = \varphi \text{ on } \partial_p Q \tag{1.2}$$

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where $Q = \Omega \times (0,T], \Omega$ is a bounded convex set in $R^n, \partial_p Q$ denotes the parabolic boundary of $Q, D_x^2 u$ is the Hessian matrix of function u, f is a nonnegative measurable function.

In 1987, by the method of continuity, Wang Guanglie [3] proved the existence and uniqueness of classical solution of (1.1) (1.2) under the assumptions that the data of the problem are relatively smooth and satisfy some compatibility conditions. To obtain better results under weaker conditions, it is needed to define generalized solution inspired from Aleksandrov's elliptic case [1].

In [2], N.V. Krylov gave a kind of generalized solution obtained from a sequence of convex functions defined on Ω. In a sense, it is an approximate generalized solution which is not perfect comparing with the elliptic case. In 1992, J.L. Spiliotis [4] proved the existence of another kind of generalized solution of (1.1) and $u = 0$ on $\partial_p Q$ with probability method, where he supposed that $\Omega \subset R^n$ is a uniformly convex open set, $\partial\Omega \in C^2, f \in C(\bar{\Omega}\times[0,\infty)), f \geq 0$ and $f = 0$ on $(\partial\Omega\times[0,\infty))\cup(\bar{\Omega}\times[T,\infty)), D_t f, D_{tt} f \in C(\bar{\Omega}\times[0,\infty))$. In [4], the author specifically pointed out that the uniqueness and regularity of his generalized solution remained open. In 1993, Wang Rouhuai and Wang Guanglie [5] gave the definition of the following generalized solution:

The Legendre transformation generated by $u(x,t)$ is:

$$\mathcal{L}_u : (x,t) \in Q \to (p,h) \in R^n \times R$$
$$p \in \nabla u(x,t), \quad h = p \cdot x - u(x,t)$$

where $u(x,t) \in C(Q)$ is a convex-monotone function, i.e. u is convex in x and nonincreasing in t.

Definition 1.1 *A convex-monotone function $u \in C(Q)$ is said to be a generalized solution of* (1.1), *if the Radon measure in Q defined by*

$$\omega_u(E) = |\mathcal{L}_u(E)| \text{ for Borel set } E \text{ of } Q$$

is absolutely continuous and its Radon-Nikedym derivative is equal to f in Q. $u \in C(\bar{Q})$ is said to be a generalized solution of (1.1) (1.2) *if it is a generalized solution of* (1.1) *and $u = \varphi$ on $\partial_p Q$.*

In [5], they first proved the equivalence of generalized solution and viscosity solution when f is continuous and then get the existence and uniqueness of viscosity solution on condition that Ω is a C^2 bounded convex domain in R^n, f is a positive continuous function on $\bar{Q}, \varphi$ is a continuous function, convex in x and non-increasing in t in a neighbourhood of $\bar{Q}$.

By a completely different way from that of [5], we obtain the results of this paper:

Theorem 1.1 *Ω is a bounded convex set in R^n, f is a nonnegative bounded measurable function defined on $Q, \varphi \in C(\partial_p Q), \varphi(x,0)$ is a convex function in $\bar{\Omega}, \forall x_0 \in \partial\Omega, \varphi(x_0,t) \in C^\alpha([0,T])$, if there exists a convex-monotone function u_φ such that for*