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STATIC AND DYNAMIC THEORIES OF LIQUID CRYSTALS

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Abstract The study of liquid crystals gives rise to many fascinating but difficult mathematical problems. The purpose of this paper is to briefly summarize some recent advances, as well as to describe the present state of art of the theory of liquid crystals. For the static theory, we emphasis on the theory of defects and the theory of Smectic A materials. We will also study the Ericksen-Leslie theory for the liquid crystal flow. The well-posedness as well as the motion of the defects will be discussed.
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1. Introduction

Liquid crystal states are the intermediate phases between the solids and the isotropic fluids. The existence of liquid crystals has been known for more than a century, although it is relatively recent that technological and industrial applications have made them familiar to the general public.

In classical literature, they are divided into three types, nematics, cholesterices, and smectics, the latter now being divided into subtypes. The commonly employed mathematical approaches to treat the equilibrium phenomena for nematic materials are, essentially, those proposed by Oseen (1925) and Frank (1955). It was obtained by straightforward macroscopic reasoning, and the theory is often referred to as the Oseen-Frank theory. For general liquid crystal materials, one often uses the Landau-de Gennes theory which involves the orientational order parameter tensor.

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The theory for nematic flow problems was developed by Ericksen and Leslie in the period from 1958 to 1968. It reduces to that of Oseen-Frank in the static case. The books [1] and [2] give many detailed discussions and references on the physics background of these problems. We also note the survey articles by Ericksen [3] and Leslie [4] which concentrated on continuum theories used for statics and flow problems are rather comprehensive. Recently, interest in and information about various special type of liquid crystals, especially those polymeric material has increased and has attracted much attention and research[5,6]. These materials are important in displays devices [7], optical switches [8] and even many applications in biotechnologies.

One of the recent developments of the display technology is the usage of polymer dispersed liquid crystals. Among many issues, one is to understand the liquid crystal droplets under different boundary conditions. Some commercial and industrial applications of such studies were discussed in an article by Doane [9]. Theoretically it involves the study of various phase transitions, the dynamics of topological defects within droplets, the shapes of droplets and the structure of defects under the influences of external fields, anchoring conditions at free interfaces and various material constants. See, for example [10–15].

Smectic materials are the phases between nematic and solids. The characteristic properties of equal distance between layers gives it distinct, however very interesting configurations. The commonly known one is the focal conic structure (see [2], [16] and their references). This was studied by Friedel [17] in 1920's. It is closely related to the Dupin cyclides and Willmore's problem in differential geometry. Also, near the phase transition between smectic A and nematic, some interesting mixed region appears, among them, the twist grain boundaries (TGB) and uniformly twist grain boundaries (USTB).

Since 1985, mathematicians started to make notable contributions, particularly in the area of the theory of defects. However, most of the work is on only the uniaxial nematics (and a few cases, cholestericеs). We refer to survey articles by Hardt-Kinderlehrer [18], J.M. Coron [19] and Lin [20] for some earlier developments (before 1990). The book by E. Virga [3] and the lecture notes by H. Brezis [21] can provide more detailed exposition of the theory. These mathematical developments and other physical motivations lead to some new theory of liquid crystals proposed by J. Ericksen [22] and MacMillan [23]. They provide many more challenging mathematical problems.

In this paper, we should summarize some recent advances of the theory as well as many unsolved problems. The paper is divided into four main sections (sections 2–5). Section 2 devoted to the introduction of the hydrostatic theory. We shall start with some general physics backgrounds of liquid crystals. After a brief discussion on liquid crystal phases, we will concentrate mainly on the orientation variables. These are kinematics variable and will be our major concern. After introducing the classical order parameter theory of Landau and de Gennes, we shown that Ericksen's model (cf. [22]) can be viewed as that of Landau-de Gennes model restricted to the uniaxial materials,