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REMARKS ON THE SHAPE OF LEAST-ENERGY SOLUTIONS TO A SEMILINEAR DIRICHLET PROBLEM

Guo Zongming
(Department of Mathematics, Henan Normal University, Xinxiang 453002, China)

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Abstract Structure of least-energy solutions to singularly perturbed semilinear Dirichlet problem $\varepsilon^2\Delta u - u^\alpha + g(u) = 0$ in $\Omega, u = 0$ on $\partial\Omega, \Omega \subset \mathbf{R}^N$ a bounded smooth domain, is precisely studied as $\varepsilon \to 0^+$, for $0 < \alpha < 1$ and a superlinear, subcritical nonlinearity $g(u)$. It is shown that there are many least-energy solutions for the problem and they are spike-layer solutions. Moreover, the measure of each spike-layer is estimated as $\varepsilon \to 0^+$.

Key Words Least-energy solutions; spike-layer solutions; singularly perturbed semilinear Dirichlet problem; nontrivial nonnegative solutions.

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1. Introduction

In this paper, we shall study least-energy solutions of the following singularly perturbed semilinear Dirichlet problem

$$\varepsilon^2\Delta u - u^\alpha + u^p = 0, \quad u \geq 0, \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N(N \geq 2), \varepsilon > 0$ is a constant, $0 < \alpha < 1, p$ satisfies $1 < p < \dfrac{N+2}{N-2}$ for $N \geq 3$ and $1 < p < \infty$ for $N = 2$. We are especially interested in the properties of the solutions as ε tends to 0. In particular, we shall establish the existence of least-energy solutions to (1.1), and show that they are *spike-layer* solutions. We also determine the location of the peak as well as the profile of the spike.

The equation (1.1) with $\alpha = 1$ is known as the stationary equation of the Keller-Segal system in chemotaxis (see [1]). It can also be seen as the limiting stationary equatoin of the so-called Gierer-Meinhardt system in biological pattern formation, see [2] for more details. In the pioneering papers of [1,3,4], Lin, Ni and Takagi established the existence of least-energy solutions to the problem

$$\varepsilon^2\Delta u - u + u^p = 0, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1.2}$$

and showed that for ε sufficiently small the least-energy solution has only one local maximum point P_ε and $P_\varepsilon \in \partial\Omega$. Moreover, $H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\varepsilon \to 0$, where $H(P)$ is the mean curvature of P at $\partial\Omega$. Note that such results also hold for more general nonlinearities than that of (1.2) (see [3,4]). Some further results for (1.2) can be found in [2,5,6] and the references therein. In [7], Ni and Wei established the existence of least-energy solutions to (1.1) with $\alpha = 1$. They obtained that for ε sufficiently small, the least-energy solution u_ε of (1.1) (with $\alpha = 1$) has at most one local maximum and it is achieved at exactly one point $P_\varepsilon \in \Omega$. Moreover, $u_\varepsilon(\cdot + P_\varepsilon) \to 0$ in $C^1_{\text{loc}}(\Omega - P_\varepsilon\backslash\{0\})$ where $\Omega - P_\varepsilon = \{x - P_\varepsilon | x \in \Omega\}$,

$$d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega) \text{ as } \varepsilon \to 0$$

To obtain all the mentioned results above, the authors used the fact that the problem

$$\Delta w - w + w^p = 0, \quad w > 0 \text{ in } \mathrm{R}^N, \ w(z) \to 0 \text{ as } |z| \to \infty \tag{1.3}$$

has a unique positive (radial) solution $w(|z|)$ which decays exponentially as $|z| \to \infty$. Meanwhile, in [8,9], the authors studied the problem

$$-\varepsilon^2\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.4}$$

with $f \in C^{1+\sigma}(0,\infty) \cap C^0([0,\infty))(0 < \sigma < 1)$, $f(0) = 0$, $f'(0) = -m < 0$ and f changing sign many times in $(0,\infty)$. When Ω is a convex domain, they found a positive small solution u_ε of (1.4), which has properties similar to that of the least-energy solution obtained in [7]. Further results for (1.4) can also be found in [10–14] and the references therein.

In this paper we are mainly interested in the properties of least-energy solutions to the problem

$$\varepsilon^2\Delta u - u^\alpha + g(u) = 0, \ u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.5}$$

where $0 < \alpha < 1$ and $g(s)$ satisfies the conditions similar to that in [3,4,7]. That is, $g : \mathrm{R} \to \mathrm{R}$ is of class $C^1(\mathrm{R})$ and satisfies the following conditions:

(g_1) $g(s) \equiv 0$ for $s \leq 0$;

(g_2) $g(s)/s$ is increasing for $s > 0$ and $\lim_{s\to+\infty} g(s)/s = +\infty$, while $g(s) = O(t^\beta)$ as $t \to 0$ with $\beta > 1$;

(g_3) $g(s) = O(s^p)$ as $s \to +\infty$, where $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < +\infty$ if $N = 2$;

(g_4) If $G(s) = \int_0^s g(t)dt$, then there exists a constant $\theta \in (0, 1/2)$ such that $G(s) \leq \theta s g(s)$ for $s \geq 0$.

Define $f(s) := g(s) - s^\alpha$ and $F(s) = \int_0^s f(t)dt$. It is easily seen that f satisfies that $f(0) = 0$ and $\lim_{s\to 0^+} f'(s) = -\infty$. Moreover, $\int_0^s |F(t)|^{-1/2}dt < \infty$ for any $s > 0$.