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THE ISOENERGY INEQUALITY FOR HARMONIC MAPS FROM ROTATIONAL SYMMETRIC MANIFOLDS

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Abstract Let u be a harmonic map from a rotational symmetric manifold M and B a unit ball in M, let $E(u|_B)$ be the energy of the map $u|_B$ and $E(u|_{\partial B})$ the energy of the map $u|_{\partial B}$, then we obtain the relationship which is called the isoenergy inequality between $E(u|_B)$ and $E(u|_{\partial B})$.

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1. Introduction

Suppose that M and N are two Riemannian manifolds of dimmensions m and n respectively, and that $u : M \to N$ is a harmonic map which is a solution of the Euler-Langrange equation of the Dirichlet integral

$$E(u) = \int_M |\nabla u|^2 dv.$$

Let $M = R^m$, and B a unit ball in R^m. We define $E(u|_B)$ and $E(u|_{\partial B})$ to be the energy of the map u and the energy of the restriction of u to ∂B respectively. Choe([1]) obtained the relationship between $E(u|_B)$ and $E(u|_{\partial B})$ which is called the isoenergy inequality.

If N is nonpositively curved, then

$$(m-1)E(u|_B) \leq E(u|_{\partial B})$$

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and the equality holds when $N = R^n, u$ is a linear map. If N is any Riemann manifold of dimension ≥ 3 and u is a stationary harmonic map, then

$$(m-2)E(u|_B) \leq E(u|_{\partial B})$$

and the equality holds if $N = S^{m-1} \subset R^m, u(x) = x/|x|$.

In this paper, we consider the relationship between $E(u|_B)$ and $E(u|_{\partial B})$ when M is a rotational symmetric manifold (see [1]). We first derive several monotonicity formulas for harmonic maps from rotational symmetric manifolds by the method used in [1] and [2]. Using these formulas we get several isoenergy inequalities which generalize Choe's result in [3]. Let $M(m \geq 3)$ be a rotational symmetric manifold, i.e., $M = (R^m, ds^2)$, where $ds^2 = dr^2 + f^2(r)d\theta^2, f(r) > 0$ for $r > 0, f'(0) = 1$ and $d\theta^2$ is the standard metric on S^{m-1}. Let $u : M \to N$ be a stationary harmonic map. We prove the following results

(1) If M has the nonpositive radical sectional curvature, then

$$(m-2)E(u|_B) \leq f(1)E(u|_{\partial B}).$$

In particular, If $f(r) = \sinh r$, i.e., M is a space form with the constant curvature -1, then

$$(m-2)E(u|_B) \leq \left(\frac{e^2-1}{2e}\right)E(u|_{\partial B}).$$

If M has the nonnegative radical sectional curvature and $f'(1) > 0$, then

$$f'(1)(m-2)E(u|_B) \leq f(1)E(u|_{\partial B}).$$

In particular, If $f(r) = \sin r$, i.e., M is a space form with the constant curvature 1, then

$$(m-2)E(u|_B) \leq (\tan 1)E(u|_{\partial B}).$$

(2) If M has the nonpositive radical sectional curvature, then

$$f^{m-3}(1)E(u|_B) \leq E(u|_{\partial B}) \int_0^1 f^{m-3}(r)dr.$$

In the case that $f(r) = r$, i.e., $M = R^m$, we reprove Choe's results in [3].

2. Monotonicity Formulas

Let $u : M \to N$ be a weakly harmonic map, u is called stationary, if for any smooth vector field X with compact support in M, $\{\Phi_s\}$ is the 1-parameter family of transforms of M generated by X, then its energy is critical with respect to the domain variations $u \circ \Phi_s$, i.e., $\frac{d}{ds}E(u \circ \Phi_s)|_{s=0} = 0$. It is proved in [2] (also see [4]) that

$$\frac{d}{ds}E(u \circ \Phi_s)|_{s=0} = -\int_M [|\nabla u|^2 \mathrm{div}(X) - 2\sum_{i=1}^m \langle du(\nabla_i X), du\left(\frac{\partial}{\partial x^i}\right)\rangle]dV, \tag{2.1}$$