J. Partial Diff. Eqs. **15**(2002), 72–82
©International Academic Publishers Vol.15 No.2

UNIQUENESS THEOREM OF THE REGULARIZABLE RADIAL GINZBURG-LANDAU TYPE MINIMIZERS

Lei Yutian
(Depart. of Math., Jilin University, Changchun 130012;
Depart. of Math., Suzhou University, Suzhou 215006, China)
(E-mail: lythxl@163.com)
(Received Oct. 19, 2001; revised Mar. 10, 2002)

Abstract The author proves the uniqueness of the regularizable radial minimizers of a Ginzburg-Landau type functional in the case $n-1<p<n$,and the location of the zeros of the regularizable radial minimizers of this functional is discussed.

Key Words Regularizable minimizer; radial minimizer; Ginzburg-Landau type functional.

2000 MR Subject Classification 35B25, 35J70.

Chinese Library Classification O175.2.

1. Introduction

Let $n \geq 2, B=\{x \in R^n; |x|<1\}, g(x)=x$ on ∂B. Consider the minimizers of the Ginzburg-Landau-type functional

$$E_\varepsilon(u,B)=\frac{1}{p}\int_B |\nabla u|^p+\frac{1}{4\varepsilon^p}\int_B (1-|u|^2)^2, \quad (n-1<p<n)$$

on the class functions

$$W=\{u(x)=f(r)\frac{x}{|x|} \in W^{1,p}(B,R^n); f(1)=1, r=|x|\}.$$

These minimizers u_ε are called *radial minimizers.*

Many papers stated the asymptotic behaviour of u_ε as $\varepsilon \to 0$ and the uniqueness of these minimizers. When $p=n=2$, the asymptotics of u_ε were well-studied by [1] and [2], and the uniqueness of u_ε was proved in [3]. Some other related works can be seen in [4] and [5] etc. When $p=n>2$ and $p>n\geq 2$, the asymptotics of u_ε were presented in [6] and [7], respectively.

Denote u_ε^τ as the minimizers of the regularized functional

$$E_\varepsilon^\tau(u,B)=\frac{1}{p}\int_B (|\nabla u|^2+\tau)^{p/2}+\frac{1}{4\varepsilon^p}\int_B (1-|u|^2)^2, \quad \tau \in (0,1)$$

on W. As $\tau \to 0$,

$$u_\varepsilon^\tau \to \tilde{u}_\varepsilon, \quad in \quad W^{1,p}(B, R^n), \tag{1.1}$$

and the $\tilde{u}_\varepsilon$ are also the minimizers of $E_\varepsilon(u, B)$ on W, which are called *the regularizable radial minimizers* of $E_\varepsilon(u, B)$ (see [8]). Applying the weakly low semicontuity we can derive easily that as $\varepsilon, \tau \to 0$,

$$u_\varepsilon^\tau \to \frac{x}{|x|}, \quad in \quad W^{1,p}(B, R^n). \tag{1.2}$$

Now, we state our main conclusion.

Theorem 1.1 *Assume $n-1 < p < n$. Then there exists a small positive constant ε_0 such that for any given $\varepsilon \in (0, \varepsilon_0)$, the regularizable radial minimizers $\tilde{u}_\varepsilon$ of $E_\varepsilon(u, B)$ are unique on W.*

Some basic properties of minimizers are given in Section 2. The main purpose of Section 3 is to prove that for any radial minimizer u_ε of $E_\varepsilon(u, B)$ and any given $\eta \in (0, 1/2)$ there exists a constant $h(\eta) > 0$ such that

$$Z_\varepsilon = \{x \in B; |u_\varepsilon(x)| < 1 - 2\eta\} \subset B(0, h\varepsilon) = \{x \in R^n; |x| < h\varepsilon\}.$$

This is Theorem 3.6 which implies, in particular, that the zeroes of u_ε are contained in $B(0, h\varepsilon)$. Based on this result, we may prove Theorem 1.1 in Section 4.

2. Preliminaries

In polar coordinates, for $u(x) = f(r)\frac{x}{|x|}$ we have

$$|\nabla u| = (f_r^2 + (n-1)r^{-2}f^2)^{1/2}, \quad \int_B |u|^p = |S^{n-1}| \int_0^1 r^{n-1}|f|^p\, dr,$$

$$\int_B |\nabla u|^p = |S^{n-1}| \int_0^1 r^{n-1}(f_r^2 + (n-1)r^{-2}f^2)^{p/2}\, dr.$$

It is easily seen that $f(r)\frac{x}{|x|} \in W^{1,p}(B, R^n)$ implies $f(r)r^{\frac{n-1}{p}-1}, f_r(r)r^{\frac{n-1}{p}} \in L^p(0,1)$. Conversely, if $f(r) \in W^{1,p}_{\text{loc}}(0,1]$, $f(r)r^{\frac{n-1}{p}-1}$, $f_r(r)r^{\frac{n-1}{p}} \in L^p(0,1)$, then $f(r)\frac{x}{|x|} \in W^{1,p}(B, R^n)$. Thus if we denote

$$V = \{f \in W^{1,p}_{\text{loc}}(0,1]; \quad r^{\frac{n-1}{p}} f_r \in L^p(0,1),$$
$$r^{(n-1-p)/p} f \in L^p(0,1), f(1) = 1\},$$

then $V = \{f(r); u(x) = f(r)\frac{x}{|x|} \in W\}$.

Substituting $u(x) = f(r)\frac{x}{|x|} \in W$ into $E_\varepsilon(u, B)(E_\varepsilon^\tau(u, B))$, we obtain

$$E_\varepsilon(u, B) = |S^{n-1}|E_\varepsilon(f) \quad (E_\varepsilon^\tau(u, B) = |S^{n-1}|E_\varepsilon^\tau(f))$$