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STARSHAPED COMPACT HYPERSURFACES WITH PRESCRIBED M-TH MEAN CURVATURE IN ELLIPTIC SPACE

Li Yanyan[1]
(Rutgers University, New Brunswick, New Jersey 08903 USA)
Vladimir I. Oliker[2]
(Emory University, Atlanta, Georgia USA)
[2] Supported by Emory University Research Committee and Emory-TU Berlin Exchange Program. (E-mail: yuan_guangwei@mail.iapcm.ac.cn)
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Abstract We consider the problem of finding a compact starshaped hypresurface in a space form for which the normalized m-th elementary symmetric function of principal curvatures is a prescribed function. In this paper the conditions for the existence of at least one solution to a nonlinear second order elliptic equation of that problem are established in case of a space form with positive sectional curvature.

Key Words Nonlinear elliptic equations; Spaces of constant sectional curvature; Hypersurfaces with prescribed curvature functions. .

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1. Introduction

Let $\mathcal{R}^{n+1}(1)$, $n \geq 2$, be a space form of sectional curvature 1 and m an integer, $1 \leq m \leq n$. In this paper we establish the conditions for the existence of a smooth hypersurface M in $\mathcal{R}^{n+1}(1)$ which is starshaped relative to some point $\mathbf{O}$ and whose m-th mean curvature $H_m = \psi_{|_M}$, where ψ is a given function in $\mathcal{R}^{n+1}(1)$. Here, by the m-th mean curvature we understand the normalized elementary symmetric function of order m of principal curvatures $\lambda_1, ..., \lambda_n$ of M, that is,

$$H_m = \frac{1}{\binom{n}{m}} \sum_{i_1 < ... < i_n} \lambda_{i_1} \cdots \lambda_{i_m}.$$

The proof of the main result uses a priori estimates obtained in preceding paper [1] and degree theory for nonlinear elliptic partial differential equations developed by Yan Yan Li [6]. We refer the reader to [1] for the introductory material, including derivation of the required partial diffrential equations, and some history of the problem.

We now state the main result of this paper. First, we describe in a convenient form the Riemannian space $\mathcal{R}^{n+1}(1)$. Let S^{n+1} be a unit sphere in Euclidean space R^{n+2} and h the standard metric on S^{n+1} induced from R^{n+2}. Let $\mathbf{O}$ be a point in S^{n+1}, S^{n+1}_+ the open hemisphere with the pole $\mathbf{O}$, and $T_{\mathbf{O}}$ the hyperplane tangent to S^{n+1} at $\mathbf{O}$. In a natural way $T_{\mathbf{O}}$ can be identified with the usual Euclidean space R^{n+1} with a Cartesian coordinate system $x = (x_1, ..., x_{n+1})$ with origin at $\mathbf{O}$. Using the inverse of the exponential map from $T_{\mathbf{O}}$ to S^{n+1}_+, we may pull the metric h from S^{n+1}_+ to an open ball $x_1^2 + ... + x_{n+1}^2 < \pi/2$ $(= B^{n+1})$ in $T_{\mathbf{O}}$ with center at $\mathbf{O}$. The space (B^{n+1}, h) is the $\mathcal{R}^{n+1}(1)$. Obviously, it is isometric to S^{n+1}_+.

Introduce in $\mathcal{R}^{n+1}(1)$ polar coordinates (u, ρ), where for a point $x \in \mathcal{R}^{n+1}(1)$ ρ is the geodesic distance from $\mathbf{O}$ to x and u is a point on a standard unit sphere S^n in R^{n+1} centered at $\mathbf{O}$ defining the direction of the geodesic from $\mathbf{O}$ to x.

The metric h in these coordinates is given by

$$h = d\rho^2 + \sin^2 \rho e, \quad 0 \leq \rho < \pi/2, \tag{1}$$

where e is the standard metric on the unit sphere S^n induced from R^{n+1}.

We consider smooth hypersurfaces in $\mathcal{R}^{n+1}(1)$ which are starshaped relative to the origin $\mathbf{O}$ and do not pass through $\mathbf{O}$, that is, such hypersurfaces are radial graphs over the sphere S^n in $\mathcal{R}^{n+1}(1)$ of positive smooth functions $z(u), u \in S^n$.

Theorem 1.1 *Let $1 \leq m \leq n$ and $\psi(x)$ a positive C^∞ function in the annulus $\bar{\Omega} \subset \mathcal{R}^{n+1}(1)$, $\bar{\Omega}:\ u \in S^n,\ \ \rho \in [R_1, R_2],\ 0 < R_1 < R_2 < \pi/2$. Suppose ψ satisfies the conditions:*

$$\psi(u, R_1) \geq \cot^m R_1 \text{ for } u \in S^n, \tag{2}$$

$$\psi(u, R_2) \leq \cot^m R_2 \text{ for } \ u \in S^n, \tag{3}$$

and

$$\frac{\partial}{\partial \rho}\left[\psi(u, \rho) \cot^{-m} \rho\right] \leq 0 \quad \text{for all } u \in S^n \ \text{ and } \rho \in [R_1, R_2]. \tag{4}$$

Then there exists a closed, C^∞, embedded hypersurface M in $\mathcal{R}^{n+1}(1)$, $M \subset \Omega$, which is a radial graph over S^n of a function z and

$$H_m(\lambda_1(z(u)), ..., \lambda_n(z(u))) = \psi(u, z(u)) \quad \text{for all } \ u \in S^n. \tag{5}$$

This theorem extends to an arbitrary $m, 1 \leq m \leq n$, the analogous result established by Oliker in [8] for $m = n$. In Euclidean space an analogous result for functions generalizing elementary symmetric functions of principal curvatures was established by Caffarelli, Nirenberg and Spruck [3]. It should be noted that in contrast with the cases studied in [3] and [8] where the usual continuity method was applied to prove existence, we have to use here the special degree theory developed in [6]. The reason for this is that the continuity method requires (among other things) that the corresponding linearized equation be invertible on any admissible solution and this result is not available in