J. Partial Diff. Eqs. **15**(2002), 13–27
©International Academic Publishers Vol.15 No.4

LOCALLY MINIMIZING SMOOTH HARMONIC MAPS FROM ASYMPTOTICALLY FLAT MANIFOLDS INTO SPHERES

Yuan Yu
(Institute of Mathematics, Academia Sinica, Beijing 100080, China)
(E-mail: yuan@math.washington.edu)
(Received Mar. 1, 2001)

Abstract By minimizing the so–called relative energy, we show that there exists a family of locally minimizing smooth harmonic maps from asymptotically flat manifolds into the standard sphere.
Key Words Harmonic map; Hardy inequality.
2000 MR Subject Classification 58E20.
Chinese Library Classification O175.29, O175.26.

1. Introduction

The existence and regularity of harmonic maps from noncompact domain with infinite energy are of interest, however we are short of this kind of examples. It was shown by Aviles–Choi–Micallef [1] that if M is complete and simply connected with sectional curvature K_M satisfying

$$-b^2 \leq K_M \leq -a^2 < 0, \tag{1.1}$$

then one can solve the Dirichlet problem with boundary at infinity for harmonic maps from M into $\overline{B_\tau(P)}$, which is a geodesic ball in the target manifolds N, where $\tau < \min\{\frac{\pi}{2}\sqrt{K}$, injectivity radius of N at $P\}, K \geq 0$ is an upper bound for the sectional curvature of N. Using heat flow and elliptic methods respectively, P.Li–L.Tam [2] and Ding–Wang [3] got some interesting existence results concerning proper harmonic maps from complete noncompact manifolds into nonpositive curvature targets.

Recently, Ding [4] introduced a so–called relative energy method for obtaining locally energy minimizing harmonic maps between noncompact manifolds. He proved that if M is simply connected with sectional curvature satisfying (1.1), then the Dirichlet problem with $C^\alpha(\alpha > 0)$ boundary data at infinity for harmonic maps from M into a compact target has a locally minimizing solution, which is smooth near infinity.

We would like to point out that the domain in the above cases has either negative curvature or positive first eigenvalue.

In this paper we discuss harmonic maps from asymptotically flat manifolds into spheres. Now that we deal with harmonic maps having infinite energy, we shall try to minimize the relative energy E_ϕ (cf. [4]) with respect to $\phi \in C^2(M^n, S^n)$ so that we can get a weakly harmonic map. Here we take a rotationally symmetric harmonic map from R^n into S^n as the approximate map ϕ near the infinity of M (cf. [5], [6]). By Hardy inequality (cf. [7], Theorem 330)

$$\inf_{u\in Y(R^n)} \frac{\displaystyle\int_{R^n} |\bigtriangledown u|^2}{\displaystyle\int_{R^n} \frac{u^2}{|x|^2}} = \left(\frac{n-2}{2}\right)^2 \tag{1.2}$$

where $Y(R^n) = \{u \in W^{1,2}_{loc}(R^n) : \|u\|_Y^2 = \int_{R^n} |\bigtriangledown u|^2 dx + \int_{R^n} \frac{u^2}{|x|^2} dx < +\infty\}$, we overcome the difficulty in proving that E_ϕ satisfies the coercive condition for $n \geq 7$. We note that in [4], Ding assumed $\lambda_1(M) > 0$ and the tension field $T(\phi) \in L^2(M)$ then used Poincaré inequality to prove the coercive condition for E_ϕ . Furthermore, we prove that $Im(f)$, the image of minimizing f is contained in the open upper hemisphere, hence the regularity of f follows. Since $Im(f)$ can not be contained in any compact set of S^n_+, the methods of [1] fail in our case. The following is our main result.

Main Theorem *Let $(M^n, g), n \geq 7$, be a complete Riemannian manifold having finite asymptotically flat ends with the least order $k > n-2$, then there exists a sequence of locally minimizing smooth harmonic maps from M into S^n.*

Remark 1.1 For $n = 2$, the Hardy inequality is invalid, for $3 \leq n \leq 6$, the bad term in E_ϕ can not be controlled since ϕ oscillates near infinite and (3.12) does not hold, so we fail to have the coercive condition for E_ϕ.

Remark 1.2 We conjecture that there is no nonconstant locally minimizing smooth harmonic maps from M^n into S^n for $2 \leq n \leq 6$. It is true in the case that M^n coincides with R^n (cf. [6]).

In the next section, we first recall the concept of harmonic maps from noncompact domain and describe the relative energy method. The main theorem will be proved in Section 3.

2. Weakly Harmonic Maps from Noncompact Domain and the Relative Energy

Let (M^m, g) be a complete noncompact Riemannian manifold and (N^n, h) be a complete Riemannian manifold. By Nash's imbedding theorem, we may embed (N, h) isometrically into a Euclidean space R^K for some large K so that N is a submanifold of R^K and h is just the induced metric.